STA 3000. Lec 14.

Lindeberg-Feller

Stoch equicontinuity.

$$\tilde{F}_n(h) = n\left(F_n(\theta^* + h/\sqrt{n}) - F_n(\theta^*)\right)$$

$$= \underbrace{n \cdot (P_n - P)\left(f_{\theta^* + h/\sqrt{n}} - f_{\theta^*}\right)}_{A_n(h)} + \underbrace{n\left(F(\theta^* + h/\sqrt{n}) - F(\theta^*)\right)}_{B_n(h).}$$

$$B_n(h) \longrightarrow \frac{1}{2} h^T \nabla^2 F(\theta^*) h.$$

Hope: $\boxed{\{A_n(h) : h \in K\} \xrightarrow{d} \{(\Sigma^*)^{1/2} Z^T h : h \in K\} \quad (*)}$.

(K compact)

$Z \sim N(0, I_d)$

$\Sigma^* = \operatorname{cov}(\nabla f_{\theta^*}(X))$

$\boxed{\begin{array}{l} f_\theta(x) \text{ differentiable at } \theta^* \quad H^* = \nabla^2 F(\theta^*) \text{ exists.} \\ |f_{\theta_1}(x) - f_{\theta_2}(x)| \le M(x) \|\theta_1 - \theta_2\|_2. \end{array}}$

If $(*)$ is true $\quad \hat{h}_n := \sqrt{n}(\hat{\theta}_n - \theta^*)$

Argmax cts mapping $\Longrightarrow \hat{h}_n \xrightarrow{d} N\left(0, (H^*)^{-1} \Sigma^* (H^*)^{-1}\right)$.

$(*) \Longrightarrow \quad A_n(h) + B_n(h) \xrightarrow{d} \frac{1}{2} h^T H^* h + (\Sigma^*)^{1/2} Z^T h.$

$$\hat{h}_n \xrightarrow{d} (H^*)^{-1} (\Sigma^*)^{1/2} Z .$$

Proof of (*).

$$\begin{array}{llll} Y_{11} & & & \\ Y_{21} & Y_{22} & & \\ Y_{31} & Y_{32} & Y_{33} & \\ & \cdots & & \end{array}$$

- Lindeberg Feller

$$Y_{ni} = f_{\theta^* + \frac{h}{\sqrt{n}}}(X_i) - f_{\theta^*}(X_i)$$

$$|Y_{ni}| \le \frac{\|h\|_2}{\sqrt{n}} \cdot M(x).$$

$\forall \varepsilon > 0,$

$$\sum_{i=1}^{n} \mathbb{E}\left[|Y_{ni}|^2 \cdot \mathbb{1}\{|Y_{ni}| > \varepsilon\} \right]$$

$$= \|h\|_2^2 \cdot \mathbb{E}\left[M(x)^2 \cdot \mathbb{1}\left\{ M(x) > \frac{\varepsilon \sqrt{n}}{\|h\|_2} \right\} \right]$$

$$\longrightarrow 0 .$$

(ii). $$\mathbb{E}\left[\sup_{\|h_1 - h_2\| \le \eta} \left| A_n(h_1) - A_n(h_2) \right| \right]$$

$$= n \cdot \mathbb{E}\left[\sup_{\|h_1 - h_2\|_2 \le \eta} \left| (P_n - P)\left(f_{\theta^* + \frac{h_1}{\sqrt{n}}} - f_{\theta^* + \frac{h_2}{\sqrt{n}}} \right) \right| \right]$$

$$\mathcal{G}_\eta := \left\{ f_{\theta^* + h_1/\sqrt{n}} - f_{\theta^* + h_2/\sqrt{n}} : \|h_1 - h_2\|_2 \le \eta,\ h_1, h_2 \in K \right\}$$

$$G(x) := \frac{M(x)\, \eta}{\sqrt{n}}, \qquad \forall g \in \mathcal{G}_\eta,\ x,\ |g(x)| \le G(x).$$

$$n \cdot \mathbb{E}\left[\sup_{g \in \mathcal{G}_\eta} \left|(P_n - P)g\right|\right] \leq C \cdot \sqrt{n} \cdot \|G\|_{L^2(P)} \cdot \int_0^1 \sqrt{\log N_{[\,]}(\delta \cdot \|G\|_{L^2(P)}, \mathcal{G}_\eta, \|\cdot\|_{L^2(P)})}\, d\delta$$

$t_1, t_2, \ldots t_N$ be ε-covering of K under $\|\cdot\|_2$,

$$\forall i,j \in [N], \quad l_{ij}(x) = f_{\theta^* + t_i/\sqrt{n}}(x) - f_{\theta^* + t_j/\sqrt{n}}(x) - \frac{2\varepsilon \cdot M(x)}{\sqrt{n}}$$

$$u_{ij}(x) = \text{-- ---- ---- ----} + \frac{2\varepsilon M(x)}{\sqrt{n}}.$$

$$\|u_{ij} - l_{ij}\|_{L^2(P)} \leq \frac{4\varepsilon}{\sqrt{n}} \cdot \|M\|_{L^2(P)}.$$

$\forall\ h, h',\quad \exists\, i,j \quad \|h - t_i\|_2 \leq \varepsilon,\ \|h' - t_j\|_2 \leq \varepsilon$

$$f_{\theta^* + h/\sqrt{n}} - f_{\theta^* + h'/\sqrt{n}} \in [l_{ij}, u_{ij}].$$

$$\|G\|_{L^2(P)} = \frac{\eta}{\sqrt{n}} \cdot \|M\|_{L^2(P)}.$$

$$\varepsilon = \frac{\eta\delta}{4}. \qquad N \leq \left(1 + \frac{\mathrm{diam}(K)}{\varepsilon}\right)^d$$

$$\#\text{brackets} \leq N^2 \leq \left(1 + \frac{4\,\mathrm{diam}(K)}{\eta\delta}\right)^{2d}.$$

$$n \cdot \mathbb{E}\left[\sup_{g \in \mathcal{G}_\eta} \left|(P_n - P)g\right|\right] \leq C \cdot \eta \cdot \|M\|_{L^2(P)} \int_0^1 \sqrt{d \cdot \log\left(\frac{1}{\delta\eta}\right)}\, d\delta$$

$$\lesssim \eta \cdot \sqrt{\log 1/\eta} \cdot \sqrt{d} \cdot \|M\|_{L^2(P)} \longrightarrow 0.$$

$$(\eta \to 0).$$

Thm (i) $f_\theta(x)$ diff at θ^*, $\Sigma^* = \mathrm{cov}(\nabla f_{\theta^*}(X))$ finite

(ii) $|f_{\theta_1}(x) - f_{\theta_2}(x)| \le M(x) \cdot \|\theta_1 - \theta_2\|_2$

(iii) $F(\theta) = \mathbb{E}[f_\theta(x)]$ twice cts diff at θ^*

$H^* \succ 0.$

(iv). $\hat{\theta}_n \xrightarrow{P} \theta^*$

Conclusion (i) $\|\hat{\theta}_n - \theta^*\|_2 = O_p\left(\frac{1}{\sqrt{n}}\right).$

(ii) $\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, (H^*)^{-1}\Sigma^*(H^*)^{-1}).$

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eg. $f_\theta(x) = |x - \theta|$ $\quad \theta^* = \mathrm{argmin}\, \mathbb{E}[|X - \theta|].$

f : 1-Lip in θ.

eg. MLE $\quad H^* = \Sigma^* = I(\theta^*).$

$\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} N(0, I(\theta^*)^{-1}).$

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