KDE.

$$\hat{p}_n(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right).$$

$\int_{\mathbb{R}} u^l K(u)\, du = 0$ for any l integer s.t. $1 \le l < \beta$.

$\int K^2 < +\infty$
$\int |K| \cdot |u|^\beta < +\infty$

$p \in \Sigma(\beta)$.

$h_n^* = n^{-\frac{1}{2\beta+1}}$ $\qquad$ $MSE(x_0) \lesssim n^{-\frac{2\beta}{2\beta+1}}$

$\forall \beta > 0$.

$$MISE = \mathbb{E}\left[\int_{\mathbb{R}} (\hat{p}_n(x) - p(x))^2 dx\right] = \mathbb{E}\left[\|\hat{p}_n - p\|_{L^2}^2\right].$$

$$= \int_{\mathbb{R}} b(x)^2 dx + \int_{\mathbb{R}} \sigma^2(x)\, dx.$$

where $b(x) = \mathbb{E}[\hat{p}_n(x)] - p(x)$

$\sigma^2(x) = \mathrm{var}(\hat{p}_n(x))$.

eg. If p is supported on $[0,1]$, $p \in \Sigma(\beta)$.

$$MISE = \int_0^1 MSE(x)\, dx \lesssim n^{-\frac{2\beta}{2\beta+1}}$$

$$S(\beta) := \left\{ f : \int_{\mathbb{R}} |f^{(\beta)}(x)|^2 dx \le 1 \right\}$$

Var:
$$\int \mathrm{var}(x)\,dx = \frac{1}{nh^2}\int \mathrm{var}\left(K\left(\frac{X_i - x}{h}\right)\right) dx$$
$$\leq \frac{1}{nh^2}\iint K^2\left(\frac{z-x}{h}\right)\cdot p(z)\,dz\,dx$$
$$= \frac{1}{nh}\int K^2(u)\,du.$$

Bias:

Detour: Generalized Minkowski Ineq. Tsybakov App. A.1

$$\|f+g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$$
$$\|g(x,u_1)+g(x,u_2)+\cdots+g(x,u_m)\|_{L^2}$$
$$\leq \|g(x,u_1)\|_{L^2} + \|g(x,u_2)\|_{L^2} + \cdots + \|g(x,u_m)\|_{L^2}$$

Thm.
$$\left\{\int\left(\int g(x,u)\,du\right)^2 dx\right\}^{1/2} \leq \int\left(\int g^2(x,u)\,dx\right)^{1/2} du$$

$$b(x) = \int K(u)\cdot\left(p(x+uh)-p(x)\right) du.$$
$$= \int K(u)\cdot\frac{(uh)^\beta}{(\beta-1)!}\int_0^1 (1-\tau)^{\beta-1}\, p^{(\beta)}(x+\tau uh)\,d\tau\,du$$

K: $(\beta-1)$-th order Kernel.

$$\int b(x)^2\,dx \le \left(\frac{h}{(\beta-1)!}\right)^{2\beta} \cdot \int \left(\int |K(u)|\cdot |u|^{\beta} \int_0^1 \left|p^{(\beta)}(x+\tau u h)\right| d\tau\, du\right)^2 dx$$

(G-Minkowski) $\le \left(\frac{h}{(\beta-1)!}\right)^{2\beta}$

$$\times \left\{ \int \left[\int K(u)^2 |u|^{2\beta} \cdot \left(\int_0^1 \left|p^{(\beta)}(x+\tau u h)\right| d\tau\right)^2 dx \right]^{1/2} du \right\}^2$$

(C-S).

$$\lesssim h^{2\beta} \left\{ \int |K(u)|\cdot |u|^{\beta} \sqrt{\int \int_0^1 \left(p^{(\beta)}(x+\tau u h)\right)^2 d\tau\, dx}\; du \right\}^2.$$

$$= \int_0^1 \left(\int p^{(\beta)}(x+\tau u h)^2 dx\right) d\tau$$

$$= \int p^{(\beta)}(x)^2 dx \le 1.$$

$$\int b(x)^2 dx \lesssim h^{2\beta} \cdot \left(\int |K(u)|\cdot |u|^{\beta} du\right)^2.$$

$$h_n^* = n^{-\frac{1}{2\beta+1}} \qquad \text{MISE} \lesssim n^{-\frac{2\beta}{2\beta+1}}$$

for $S(\beta)$.

Lack of asymptotic optimality for fixed p.

For $p \in S(2)$. $\int (p''(x))^2 dx < +\infty$. p supported on $[0,1]$.

MISE for KDE $\asymp n^{-4/5}$ using 1st-order Kernel.

$h_n^* = n^{-1/5}$ $\left(\int K(u)\cdot u\,du = 0\right)$

Thm. K be 2nd order Kernel, $\int K^2 du < +\infty$ $\int |K(u)|\cdot|u|^2 < +\infty$

$\forall \varepsilon > 0$, $h = \frac{n^{-1/5}}{\varepsilon}\cdot\int K^2(u)\,du$, then

$$\limsup_{n\to+\infty} n^{4/5}\,\mathbb{E}\left[\int(\hat{p}_n(x) - p(x))^2 dx\right] \le \varepsilon.$$

"Every function is asymptotically easier than average"

Proof: Claim the following

(1). $$\int \sigma^2(x)\,dx = \frac{1}{nh}\int K^2(u)\,du + o\left(\frac{1}{nh}\right)$$

(2). $$\int b^2(x)\,dx = \frac{h^4}{4}\left(\int u^2 K(u)\,du\right)^2\cdot\left(\int p''(x)^2 dx\right) + o(h^4).$$

$(h \to 0,\ nh \to +\infty)$.

Given (1) and (2). for K 2nd order $\int u^2 K(u)\,du = 0$

$\int b^2(x)\,dx = o(h^4)$.

$$MISE = \frac{1}{nh}\int K^2(u)\,du\cdot(1 + o(1)) + o(h^4).$$

$$h = n^{-1/5} \varepsilon^{-1} \int K^2(u)\,du \Rightarrow \limsup \text{MISE} \le \varepsilon.$$

Proof (1). $\int \sigma^2(x)\,dx = \frac{1}{nh}\int K^2(u)\,du - \frac{1}{nh^2}\int\left(\int K\left(\frac{z-x}{h}\right)p(z)dz\right)^2 dx$

$$\frac{1}{n}\int\left(\int K(u)\cdot p(x+uh)\,du\right)^2 dx \le \frac{1}{n}\left[\int |K(u)|\cdot\left(\int p^2(x+uh)\,dx\right)^{1/2} du\right]^2$$

$$= O\left(\frac{1}{n}\right)$$

Proof of (2).

$$b(x) = h^2 \int u^2 K(u)\cdot\left[\int_0^1 (1-\tau)\,p''(x+\tau uh)\,d\tau\right] du.$$

$$\tilde{b}(x) = h^2 \int u^2 K(u)\left[\int_0^1 (1-\tau)\cdot p''(x)\,d\tau\right] du.$$

$$\boxed{\int \tilde{b}(x)^2 dx = \frac{h^4}{4}\left(\int u^2 K(u)\,du\right)^2\cdot\left(\int p''(x)^2 dx\right)}$$

$$\left|\int b^2(x)\,dx - \int \tilde{b}^2(x)\,dx\right|$$

$$\le \underbrace{\left[\int \left(b(x)+\tilde{b}(x)\right)^2 dx\right]^{1/2}}_{\le O(h^2)} \cdot \underbrace{\left[\int \left(b(x)-\tilde{b}(x)\right)^2 dx\right]^{1/2}}_{\le o(h^2).}$$

(we know $\int \left\{ \int u^2 K(u)\cdot \underbrace{\left[\int_0^1 (1-\tau)\,p''(x+\tau uh)\,d\tau\right]}_{p''(x).} du \right\}^2 dx < +\infty$ $<+\infty$)

$$I(h) = \int \left(\int u^2 K(u) \cdot \left[\int_0^1 (1-\tau) \left(p''(x+\tau u h) - p''(x) \right) d\tau \right] du \right)^2 dx$$

$$\leq \left(\int u^2 |K(u)| \left[\underbrace{\int \left(\int_0^1 \left| p''(x+\tau u h) - p''(x) \right| d\tau \right)^2 dx}_{} \right]^{1/2} du \right)^2 .$$

$$\leq \int_0^1 \int \left| p''(x+\tau u h) - p''(x) \right|^2 dx \, d\tau$$

$$\leq \sup_{\tau \in [0,1]} \| p''(\cdot + \tau u h) - p'' \|_{L^2}^2$$

K has bdd support (for simplicity, c.f. Tsybakov Prop. A.1)

K supported on $[-1,1]$.

$$I(h) \leq \int u^2 \cdot |K(u)| \, du \cdot \sup_{\substack{\tau \in [0,1] \\ u \in [-1,1]}} \| p''(\cdot + \tau u h) - p'' \|_{L^2}^2$$

$p'' \in L^2[0,1]$. $\forall \varepsilon > 0$ $\exists g \in C[0,1]$, $\| p'' - g \|_{L^2} \leq \varepsilon$.

$$\| p''(\cdot + \tau u h) - p'' \|_{L^2} \leq \underbrace{\| p''(\cdot + \tau u h) - g(\cdot + \tau u h) \|_{L^2}}_{\leq \varepsilon}$$

$$+ \| g(\cdot + \tau u h) - g \|_{L^2} \quad \xrightarrow{(h \to 0)} 0$$

$$+ \underbrace{\| p'' - g \|_{L^2}}_{\leq \varepsilon}$$

$$\|p''(\cdot + \tau_n h) - p''\|_{L^2} \longrightarrow 0 \qquad (h \to 0).$$

Nonpara regression.

$$Y_i = f^*(i/n) + \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} N(0,1).$$

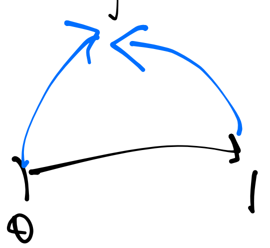
$(x_i = i/n$, equi-spaced design).

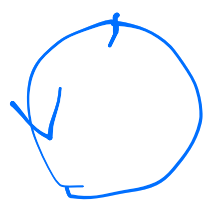
Trigonometric basis

$$\varphi_1 = 1$$
$$\varphi_{2k} = \sqrt{2}\cos(2\pi k x)$$
$(k \geq 1)$
$$\varphi_{2k+1} = \sqrt{2}\sin(2\pi k x).$$

$\{\varphi_j\}_{j\geq 1}$ orthonormal basis on $L^2[0,1]$.

f^* periodic $(f^*(0) = f^*(1))$.





(Torus).

$$W^{per}(\beta) = \left\{ f : [0,1] \to \mathbb{R},\ \|f^{(\beta)}\|_{L^2} \leq 1,\ \|f\|_{L^\infty} \leq 1,\ f^{(j)}(0) = f^{(j)}(1) \text{ for } j = 0, 1, \cdots, \beta - 1 \right\}$$

$$\Theta(\beta) = \left\{ \theta \in \ell^2(\mathbb{N}) : \sum_{j=1}^{+\infty} j^{2\beta} \cdot \theta_j^2 \leq 1 \right\}$$

Prop: $f \in W^{per}(\beta)$, then $\theta_j = \langle f, \varphi_j \rangle_{L^2}$

$\theta \in \Theta(\beta)$.

$$\widehat{f^{(l)}} = (2\pi i j)^l \cdot \hat{f}$$

Estimator $\hat{\theta}_j = \frac{1}{n} \sum_{i=1}^{n} Y_i \varphi_j(X_i) \quad \left(= \langle Y, \varphi_j \rangle_n \right)$

N to be determined.

$$\hat{f}_{n,N}(x) = \sum_{j=1}^{N} \hat{\theta}_j \varphi_j(x).$$

Key properties. $\mathbb{E}[\hat{\theta}_j] = \langle f^*, \varphi_j \rangle_n$

$$\mathrm{var}(\hat{\theta}_j) = \frac{1}{n^2} \sum_{i=1}^{n} \varphi_j^2(i/n) = \frac{1}{n}.$$