STA 3000F Lec 20

$$f = \sum_{j=1}^{+\infty} \theta_j \varphi_j \qquad f \in W^{per}(\beta) \Rightarrow \theta \in \Theta(\beta)$$

$$\left\{ \ell^2(\mathbb{N}) : \sum_{j=0}^{+\infty} j^{\beta} \theta_j^2 \le 1 \right\}.$$

$j=1,2,\dots,N$, $\quad \hat{\theta}_j = \frac{1}{n}\sum_{i=1}^{n} Y_i \varphi_j(X_i)$

$$(\|\varphi_j\|_{L^2} = 1)$$

$$\mathbb{E}[\hat{\theta}_j] = \langle f^*, \varphi_j \rangle_n.$$

$$\mathrm{Var}(\hat{\theta}_j) = \frac{1}{n^2}\sum_{1}^{n} \varphi_j(i/n)^2 = \frac{1}{n}. \qquad \|\varphi_j\|_n = 1.$$

$$\mathbb{E}\left[\int_0^1 \left(\hat{f}_n(x) - f^*(x)\right)^2 dx\right] = \sum_{j=1}^{+\infty} \mathbb{E}\left(\hat{\theta}_j - \theta_j^*\right)^2$$

$$= \sum_{j=1}^{N} \mathbb{E}\left[(\hat{\theta}_j - \theta_j^*)^2\right] + \sum_{j=N+1}^{+\infty} (\theta_j^*)^2$$

$$= \sum_{j=1}^{N} \left[\frac{1}{n} + \Big(\underbrace{\langle f^*, \varphi_j\rangle_n - \langle f^*, \varphi_j\rangle_{L^2}}_{\alpha_j}\Big)^2\right] + \sum_{j=N+1}^{+\infty} (\theta_j^*)^2$$

$$\le \frac{N}{n} + \frac{\sum_{j=N+1}^{+\infty} j^{2\beta} (\theta_j^*)^2}{N^{2\beta}} + \sum_{j=1}^{N} \alpha_j^2 .$$

$$\boxed{\frac{N}{n} + \frac{1}{N^{2\beta}}} \qquad N = n^{\frac{1}{2\beta+1}}$$

(Ignore α) $\quad \mathrm{MISE} \lesssim n^{-\frac{2\beta}{2\beta+1}}$.

Following this calculation.

$$\mathbb{E}\left[\|\hat{f}_n - f^*\|_n^2\right] \lesssim n^{-\frac{2\beta}{2\beta+1}}$$

MISE: need to bound α $\quad \left(f^* = \sum_{j=1}^{+\infty} \theta_j \varphi_j\right)$

$$\alpha_j = \sum_{i=1}^{+\infty} \theta_i^* \left(\langle \varphi_i, \varphi_j \rangle_n - \langle \varphi_i, \varphi_j \rangle_{L^2} \right). \quad (j \le N < n).$$

If $i, j \le n$. $\langle \varphi_i, \varphi_j \rangle_n = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases}$

$$\alpha_j = \sum_{i=n+1}^{+\infty} \left(\langle \varphi_i, \varphi_j \rangle_n - \langle \varphi_i, \varphi_j \rangle_{L^2} \right) \theta_i^*.$$

$$\left(\|\varphi\|_\infty \le \sqrt{2} \right) \quad \le 2 \sum_{i=n+1}^{+\infty} |\theta_i^*|$$

$$\sum_{i=n+1}^{+\infty} |\theta_i^*| \le \left\{ \sum_{i=n+1}^{+\infty} i^{2\beta} \cdot (\theta_i^*)^2 \right\}^{1/2} \cdot \left\{ \sum_{i=n+1}^{+\infty} i^{-2\beta} \right\}^{1/2}$$

$$\left(\beta > \tfrac{1}{2}\right) \quad \le C \cdot n^{\frac{1}{2} - \beta}$$

$$\text{MISE} \le \frac{N}{n} + N^{-2\beta} + N \cdot n^{1-2\beta}$$

Want to take $N = \lceil n^{\frac{1}{2\beta+1}} \rceil$ In order for $\text{MISE} \lesssim n^{-\frac{2\beta}{2\beta+1}}$ we need $N \cdot n^{1-2\beta} \lesssim n^{-\frac{2\beta}{2\beta+1}}$

which requires $\beta \geq 1$.

Remark: $\hat{f}_{n,N}$ relies on $\{\varphi_j\}_{j=1}^{+\infty}$ being orthonormal in $L^2[0,1]$.

Local poly estimator

Warmup: Nadaraya-Watson

$$\hat{f}_n(x) = \frac{\sum_{i=1}^{n} Y_i K\left(\frac{X_i - x}{h}\right)}{\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)}$$

K: some (well-behaved) Kernel.

$$W_{n,i}(x) = \frac{K\left(\frac{X_i - x}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h}\right)}.$$

Bias: $b(x_0) = \mathbb{E}[\hat{f}_n(x_0)] - f^*(x_0)$

$$= \sum_{i=1}^{n} W_{n,i}(x) \cdot \left(f^*(x_i) - f^*(x_0)\right).$$

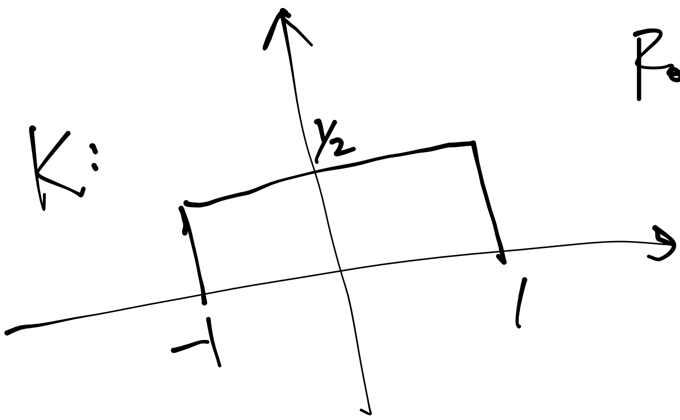
Variance: $\sigma^2(x_0) = \sum_{i=1}^{n} W_{n,i}^2(x_0)$.

Assume $f^* \in \Sigma(\beta)$. i.e. $|f^*(x) - f^*(y)| \le |x-y|^\beta$

for $\beta \in (0,1]$.

$$|b(x_0)| \le \sum_{i=1}^n |W_{n,i}(x_0)| \cdot |x_i - x_0|^\beta$$

K:



For x_i's s.t. $|x_i - x_0| > h$

$$K\left(\frac{x_i - x_0}{h}\right) = 0$$

$$\Rightarrow W_{n,i}(x_0) = 0.$$

$$|b(x_0)| \le \sum_{i=1}^n |W_{n,i}(x_0)| \cdot h^\beta = h^\beta$$

$$\sigma^2(x_0) = \sum_{i=1}^n W_{n,i}(x_0)^2 \le \max_i |W_{n,i}(x_0)| \cdot \underbrace{\sum_{i=1}^n |W_{n,i}(x_0)|}_{=1}.$$

$$W_{n,i}(x_0) = \frac{K\left(\frac{x_i - x_0}{h}\right)}{\sum_{j=1}^n K\left(\frac{x_j - x_0}{h}\right)}$$

$$\le \left|\{j : |x_j - x_0| \le h\}\right|^{-1}.$$

Equi-spaced design $\quad h > n^{-1}$.

$$\left| \{ j : |x_j - x_0| \le h \} \right| \geq nh.$$

$$MSE(x_0) \lesssim h^{2\beta} + \frac{1}{nh}$$

$$h_n^* = n^{\frac{-1}{2\beta+1}} \Longrightarrow MSE(x_0) \lesssim n^{\frac{-2\beta}{2\beta+1}}.$$