STA3000F Lecture 22

$H_1: X \sim P_1$

$H_2: X \sim P_2$

$\vdots$

$H_M: X \sim P_M$

$J \sim \text{Unif}(\{1, 2, \dots, M\})$.

$\mathbb{P}(T(X) \neq J) \geq ?$

for any estimator (test) T.

Thm (Fano)

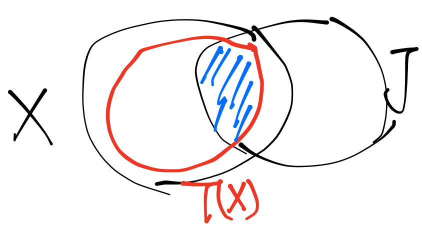
$$\mathbb{P}(T(X) \neq J) \geq 1 - \frac{I(X;J) + \log 2}{\log M}.$$

$$I(X;Y) = D_{KL}(P_{X,Y} \,\|\, P_X \times P_Y).$$

Proof of Fano:

Step I: $\underline{\underline{I(X;J) \geq I(T(X);J)}}$.

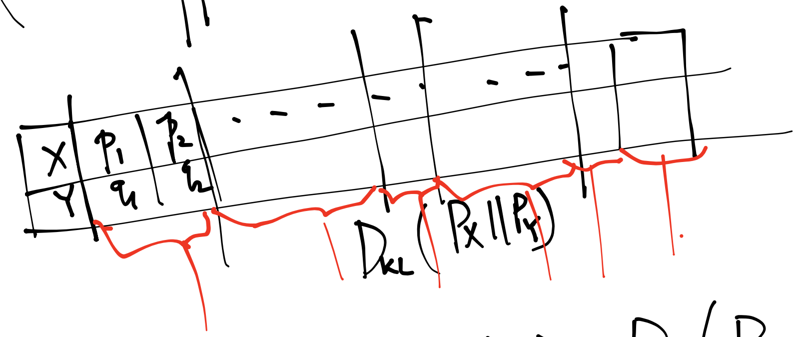
$J \to X \to T(X)$



$$I(X;J) = I(X, T(X); J) = I(T(X); J) + \underbrace{I(X; J \mid T(X))}_{\geq 0.}$$

Step II. $I(\underbrace{T(X)}_{\in\{1,2,\dots,M\}};\underbrace{J}_{\in\{1,2,\dots,M\}}) \geq ?$

$$\boxed{D_{KL}\left(P_{T(X),J} \,\middle\|\, P_{T(X)} \times \mathrm{Unif}(\{1,2,\dots,M\})\right).}$$



$$D_{KL}(P_X\|P_Y) \geq D_{KL}(P_{f(X)}\|P_{f(Y)}).$$

(Jensen's ineq)

$f(t,j) = 1\{t\neq j\}$. $\quad P_e := \mathbb{P}(T(X)\neq J)$

$$\boxed{I(T(X);J) \geq D_{KL}\left(\mathrm{Ber}(P_e)\,\middle\|\,\mathrm{Ber}\left(1-\tfrac{1}{M}\right)\right).}$$

$$= P_e \log \frac{P_e}{1-\frac{1}{M}} + (1-P_e)\log\left(\frac{1-P_e}{1/M}\right).$$

$$= -H(\mathrm{Ber}(P_e)) + \log M - P_e \log (M-1).$$

$$\boxed{H(\mathrm{Ber}(p)) = p\log 1/p + (1-p)\log 1/(1-p) \leq \log 2}$$

$$\geq -\log 2 + \log M - P_e \log M.$$

Useful fact: $\bar{P} = \frac{1}{M}\sum_{1}^{M} P_j$

$$I(X;J) = \frac{1}{M}\sum_{j=1}^{M} D_{KL}(P_j \| \bar{P})$$
$$= \frac{1}{M}\sum_{j=1}^{M} D_{KL}(P_j \| Q) - D_{KL}(Q \| \bar{P}).$$
$$\leq \frac{1}{M}\sum_{j=1}^{M} D_{KL}(P_j \| Q).$$

Corollary: If we have $\|f_i - f_j\|_{L^2} \geq 2\delta \quad \forall i \neq j.$

$$\inf_{\hat{f}} \mathbb{E}\left[\|\hat{f} - f_J\|_{L^2}^2\right] \geq \delta^2 \cdot \left[1 - \frac{I(X;J) + \log 2}{\log M}\right]$$

Proof: Given $\hat{f}$, let $T := \operatorname{argmin}_{j \in [M]} \|\hat{f} - f_j\|_{L^2}$.

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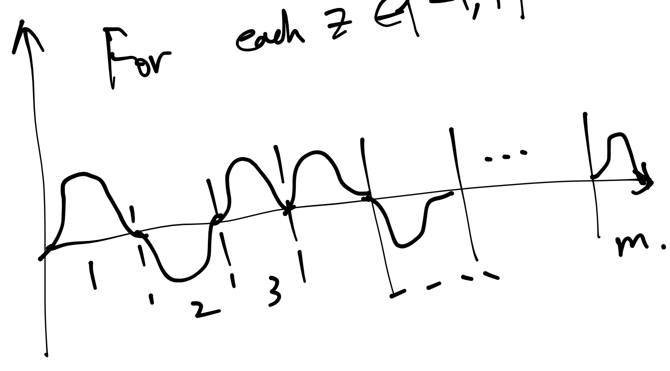
suffices to construct $f_1, f_2, \dots, f_M$

s.t. 1. $\|f_i - f_j\| \geq 2\delta \quad \forall i \neq j$

2. M large

3. $\frac{1}{M}\sum_{i=1}^{M} D_{KL}(P_j \| Q)$ small.

For each $z \in \{-1,1\}^m$



2^m functions.

Bandwidth $h = \frac{1}{2m}$ $\qquad Q = P_0$

Height $\psi = h^\beta$. $f_z \in \Sigma(\beta)$.

$$D_{KL}(P_{f_z} \| P_0) = \frac{n}{2}\|f_z\|_n^2 \le \frac{n\psi^2}{2}.$$

$$\|f_z - f_{z'}\|_{L^2} \gtrsim ? \quad (\text{for } z \neq z').$$

Gilbert-Varshamov.

Sample $z^{(1)}, z^{(2)}, \cdots, z^{(M)} \overset{iid}{\sim} \mathrm{Unif}(\{\pm 1\}^m)$.

Hoeffding : $\mathbb{P}\left(d(z^{(i)}, z^{(j)}) \le \frac{m}{2} - t\right) \le \exp\left(-\frac{2t^2}{m}\right)$. $(\forall i,j)$

$\boxed{d(x,y) = \#\text{ mismatch between } x \text{ and } y}$.

$$\mathbb{P}\left(\exists i,j,\ d(z^{(i)}, z^{(j)}) \le \frac{m}{2} - t\right) \le \binom{M}{2} \cdot e^{-\frac{2t^2}{m}}$$

Take $t = \frac{m}{4}$ and $M = \exp\left(\frac{m}{16}\right)$

$$\mathbb{P}\left(\exists i,j \in [M],\ d(z^{(i)}, z^{(j)}) \le \frac{m}{4}\right) \le \frac{1}{2}.$$

$\exists \{z^{(j)}\}_{j \in [M]}$ s.t. $\|f_{z_i} - f_{z_j}\|_{L^2} \ge \frac{\psi}{8}$.

$$M = \exp\left(\frac{m}{16}\right).$$

$$\log M \asymp m \asymp 1/h.$$

$$\inf \sup \mathbb{E}\left[\|\hat{f}-f\|_{L^2}^2\right] \gtrsim \psi^2 \cdot \left(1 - \frac{I(X;J)+\log 2}{m}\right)$$

$$I(X;J) \lesssim n\psi^2.$$

$$1/h = m \geq n\psi^2 = n \cdot h^{2\beta}$$

$$h_n = n^{\frac{-1}{2\beta+1}}. \qquad \psi = n^{\frac{-\beta}{2\beta+1}}$$

$$MISE \gtrsim n^{-\frac{2\beta}{2\beta+1}}.$$