STA3000F: Homework 1

October 2, 2023

1 Q1: Estimating a permutation

A permutation is a bijection from $\{1, 2, \cdots, n\}$ to itself. Let $\tau : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be an unknown permutation of the indices. Given a known vector $\mu \in \mathbb{R}^n$ such that $\mu_1 < \mu_2 < \cdots < \mu_n$ where μ_i is the i-th coordinate of μ. Consider independent observations

$$X_i = \mu_{\tau(i)} + \varepsilon_i, \quad \text{for } i = 1, 2, \cdots, n,$$

where $\varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Consider the loss function

$$L(\tau, d) := \sum_{1 \leq \ell, m \leq n} \mathbf{1}\big[\tau(\ell) < \tau(m), d(\ell) > d(m)\big].$$

Find a minimax estimator for τ.

2 Q2: Linear regression

Given integers $n > d$, consider a matrix X formed by a collection of known and deterministic d-dimensional vectors

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \in \mathbb{R}^{d \times n}.$$

Assuming that X has full row rank. Let $\theta \in \mathbb{R}^d$ be an unknown parameter. Let the observations be given by

$$Y_i = x_i^\top \theta + \varepsilon_i, \quad \text{for } i = 1, 2, \cdots, n,$$

where $\varepsilon_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$.

An estimator δ is said to be UMVU for a vector-valued functional g if for any other unbiased estimator $\widetilde{\delta}$, we have that

$$\mathbb{E}_\theta\big[(\widetilde{\delta} - g(\theta))(\widetilde{\delta} - g(\theta))^\top\big] \succeq \mathbb{E}_\theta\big[(\delta - g(\theta))(\delta - g(\theta))^\top\big].$$

Find a UMVU estimator for $g(\theta) = \theta$, and compute its covariance.

3 Q3: Le Cam's two-point method

Consider a simple vs. simple hypothesis testing problem: $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$. Assume that the densities (with respective to k-dimensional Lebesgue measure) p_0 and p_1 exist, and have common support.

1. For any test ϕ, show that

$$\mathbb{E}_{\theta_0}\big[\phi(X)\big] + \mathbb{E}_{\theta_1}\big[1-\phi(X)\big] \ge 1 - d_{\mathrm{TV}}(p_1, p_0),$$

   where d_{TV} denotes the total variation distance $d_{\mathrm{TV}}(p,q) := \frac{1}{2}\int |p(x)-q(x)|dx$.

2. Now we consider the parameter estimation problem. Let $g(\theta) \in \mathbb{R}$ be the functional of interest, and consider mean-squared loss. Show that

$$\inf_{\delta} \sup_{\theta \in \{\theta_0,\theta_1\}} \mathbb{E}\big[|\delta(X) - g(\theta)|^2\big] \ge \frac{1}{8}|g(\theta_0) - g(\theta_1)|^2 \cdot \Big\{1 - d_{\mathrm{TV}}(p_1, p_0)\Big\}.$$

   [Hint: consider a uniform prior distribution on $\{\theta_0, \theta_1\}$.]

4 Bonus question: sequential decision making

In the class, we work with statistical decision theory in a stochastic setting. Though we allow randomized estimators to ensure convexity, the optimal estimator is usually deterministic. However, in (adversarial) sequential decision making, randomness and convexification is crucially needed. Let us consider the following problem.

Let the action space be $\mathcal{A} = \{0,1\}$. Consider a collection of $K \ge 2$ experts. For $t = 1, 2, \cdots, T$, each expert $k \in \{1, 2, \cdots, K\}$ gives an advice $a_{k,t} \in \{0,1\}$. After observing the experts' advice, we are required to make a decision $a_t \in \{0,1\}$. After that, the ground truth $y_t \in \{0,1\}$ at t-th round is revealed, and incur the loss $L(y,a) = \mathbf{1}[y \neq a]$. A strategy π generates the action a_t based on the entire history, i.e., the experts' advice and ground truths in the past, and the experts' advice at t-th round. We use π_t to denote the probability of taking action 1 at time t. Our goal is to achieve small "regret":

$$R_n(\pi) := \sum_{t=1}^{n} L(y_t, a_t) - \min_{k \in K} \sum_{t=1}^{n} L(y_t, a_{k,t}).$$

Note that the ground truth y_t and experts advice $a_{k,t}$ do not have to follow a probabilistic model. They may be adaptively chosen in an adversarial manner. Please show the following:

- Let $K = 2$, for any deterministic strategy π, there exists an adversary's strategy, such that $R_n(\pi) \ge n/2$.
- Now let's consider randomized strategies. Given a scalar $\varepsilon > 0$, define the weights

$$w_{i,t} := \exp\left(-\varepsilon \sum_{j=1}^{t} L(y_j, a_{i,j})\right),$$

  i.e., down-weight the expert j by a factor $e^{-\varepsilon}$ for each mistake he/she makes. Consider the decision strategy

$$\pi_{t+1} = \frac{\sum_{i=1}^{K} a_{i,t+1} w_{i,t}}{\sum_{i=1}^{K} w_{i,t}}.$$

Show that the regret satisfies

$$\mathbb{E}[R_n(\pi)] \leq \frac{\log K}{\varepsilon} + \varepsilon n,$$

and by taking $\varepsilon = \sqrt{\frac{\log K}{n}}$, we have $\mathbb{E}\big[R_n(\pi)\big] \leq 2\sqrt{n \log K}$.

[Hint: study the evolution of $W_t = \sum_{i=1}^{K} w_{i,t}$.]