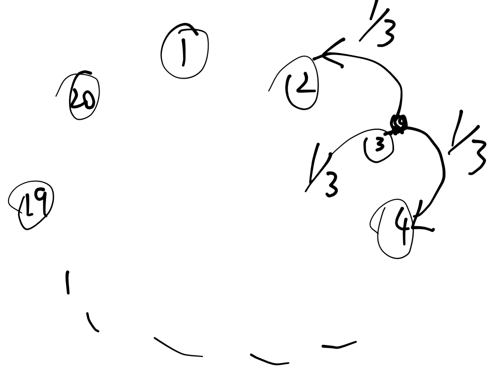
STA 447/2006 Lecture 1

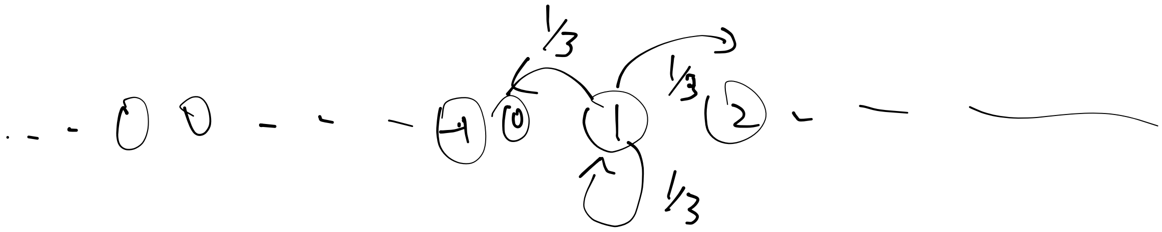
$(X_t : t \in T)$ T being a time interval

Special temporal structures

- Markov — Forgetting history — T being integers
- martingale — Fair gambling
- Continuous-time processes. (Brownian motion).

Itô's calculus.





Def <u>discrete-time</u>, <u>discrete-state</u>, <u>time-homogeneous</u>

Markov chain $(X_0, X_1, X_2, \cdots)$

(i) State space S (finite or countably infinite)

(ii) Initial distribution $(\nu_i)_{i \in S}$ $\quad \nu_i = \mathbb{P}(X_0 = i)$

(iii) Transition probabilities $(p_{ij})_{i,j \in S}$

$$p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i) = \frac{\mathbb{P}(X_{t+1} = j, X_t = i)}{\mathbb{P}(X_t = i)}.$$

eg. Frog walk

$S = \{1, 2, \cdots, 20\}$

Initial distr $\quad \nu_3 = 1 \quad \nu_j = 0$ for $j \neq 3$

Transition prob

$$p_{ij} = \begin{cases} 1/3 & |i-j| \leq 1 \text{ or } |i-j| = 19 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n)$$

$$= \mathbb{P}(X_0 = i_0) \cdot \mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdot \mathbb{P}(X_2 = i_2 \mid X_0 = i_0, X_1 = i_1)$$
$$\cdots \cdot \mathbb{P}(X_n = i_n \mid X_0 = i_0, \cdots X_{n-1} = i_{n-1})$$

$$= \nu_{i_0} \cdot p_{i_0 i_1} p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$$

Key fact $\mathbb{P}(X_n = i_n \mid X_0 = i_0, X_1 = i_1, \cdots X_{n-1} = i_{n-1}) = \mathbb{P}(X_n = i_n \mid X_{n-1} = i_{n-1})$

More examples.

eg. At time t, flip a coin

$$X_t = \# \text{heads for time } 1, 2, \cdots t$$

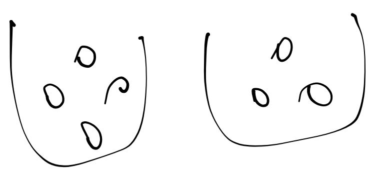
$$P_{ii} = 1/2, \quad P_{i(i+1)} = 1/2.$$

$\nu_0 = 1$

$\nu_i = 0 \ (i \neq 0)$

eg. Ehrenfest's Urn.

d balls in total



At each time, randomly select a ball, and move

$$X_n := \# \text{ balls in the left side}$$

$$S = \{0, 1, 2, \cdots, d\}$$

$$P_{i,i-1} = i/d \qquad P_{i,i+1} = 1 - i/d$$

eg. Multi-dim RW.

w.p. $\frac{1}{2d}$ move to each neighboring state