STA 447/2006 Lecture 10.

Metropolis – Hastings algorithm

Goal: generate a sample from π

$$\left(\text{eg. } \pi_i \propto \exp(-f(i)) \quad \forall i \in S\right)$$

Use a Markov transition Kernel q

If π is not stationary of q

"Adjustment" $\longrightarrow$ p converging to π.

— q: "proposal distribution"

Simpler case $q(i,j) = q(j,i)$

$$P_{ij} = \begin{cases} q(i,j) \cdot \min\left(1, \dfrac{\pi_j}{\pi_i}\right) & (j \neq i) \\ 1 - \sum_{l \neq i} P_{il} & (j = i) \end{cases}$$

Idea: accept the proposal w.p. $\min\left(1, \frac{\pi_j}{\pi_i}\right)$.

$$\pi_i P_{ij} = q(i,j) \cdot \min(\pi_i, \pi_j)$$
$$= q(j,i) \min(\pi_j, \pi_i) = \pi_j P_{ji}$$

— π is stationary of P

— Irreducibility: $i \to j$ under q

then $i \to j$ under P

— Aperiodicity: true as long as you ever reject.

$$\lim_{n \to \infty} P_{ij}^{(n)} = \pi_j$$

$\mathbb{E}_\pi[|f|] < \infty$ $$\frac{1}{n} \sum_{1}^{n} f(X_t) \longrightarrow \mathbb{E}_\pi[f(X)].$$

— Asymmetric q.

$$P_{ij} = q(i,j) \cdot \min\left[1, \frac{\pi_j q(j,i)}{\pi_i q(i,j)}\right].$$

$$\pi_i P_{ij} = \min\left[\pi_i q(i,j), \pi_j q(j,i)\right] = \pi_j P_{ji}.$$

Random Walk on graphs.

- V: vertices
- E: edges
- $w(u,v)$: weight func $\quad (w(u,v) \geq 0)$.

$(V = S)$.

Def. : (SRW on weighted graph)

$$p_{uv} = \frac{w(u,v)}{\sum_{i \in S} w(u,i)}$$

$\sum_{i \in S} w(u,i) := d(u)$

(Assume $d(u) < +\infty \quad \forall u \in S$)

Stationary measure

$$\mu(u) = d(u)$$

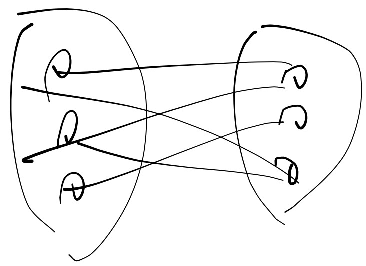
$$d(u) \cdot p_{uv} = w(u,v) = w(v,u) = d(v) \cdot p_{vu}$$

If $Z := \sum_{u \in S} d(u) < +\infty$

then $\pi_u = \frac{d(u)}{Z}$ is stationary distribution

— Irreducible : graph connected

— Aperiodicity : { bipartite graph — period 2
otherwise — period 1.



Sequence waiting time

Coin toss

HHTHTTTHH ....

"HTH": 4-th toss.

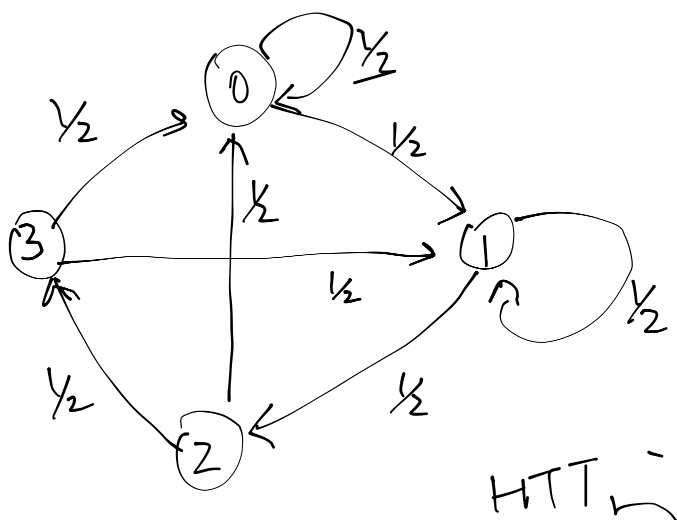
"THH" 9-th toss

$\mathbb{E}$[# tosses needed] ?

X_n := # bits in the desired sequence achieved at n-th toss.

$S = \{0, 1, 2, 3\}$

"HTH"



TH
HH - - -

HTT - - - -

$\mathbb{E}_0[\tau_3]$

$\mathbb{E}_3[\tau_3]$

$$\pi = \begin{array}{c} \begin{bmatrix} 0.3 & 0.4 & 0.2 & 0.1 \end{bmatrix} \\ \begin{matrix} \;\;0\;\; & \;\;1\;\; & \;\;2\;\; & \;\;3\;\; \end{matrix} \end{array}$$

$\mathbb{E}_3[\tau_3] = 10.$

Seq	H	T	H	T	H	
X_n	0	1	2	3	0	1

Martingales

"Fair gambling"

A sequence of r.v.s $(X_n)_{n=0}^{+\infty}$ is a martingale

if $\mathbb{E}\left[|X_{n+1}| \,\middle|\, X_1 \cdots X_n\right] < +\infty$ a.s.

and $\mathbb{E}\left[X_{n+1} \,\middle|\, X_1 \cdots X_n\right] = X_n$ $(\forall n = 0, 1, \cdots)$.

e.g. Simple random walk on $\mathbb{Z}$

e.g. $Y_1, Y_2, \cdots Y_n \overset{iid}{\sim}$ $\mathbb{E}\left[|Y_i|\right] < +\infty$.
$\mathbb{E}[Y_i] = 0$.

$X_n = \sum_{i=1}^{n} Y_i$

Notation: $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$

Information contained in $X_1 \cdots X_n$

$\mathbb{E}\left[X_{n+1} \,\middle|\, \mathcal{F}_n\right] = X_n$

Fact. $\mathbb{E}\left[X_n \,\middle|\, \mathcal{F}_m\right] = X_m$

$(0 \le m < n)$

$\mathbb{E}\left[X_{n+2} \,\middle|\, \mathcal{F}_n\right] = \mathbb{E}\left[\mathbb{E}\left[X_{n+2} \,\middle|\, \mathcal{F}_{n+1}\right] \,\middle|\, \mathcal{F}_n\right] = \mathbb{E}\left[X_{n+1} \,\middle|\, \mathcal{F}_n\right]$
$= X_n$

eg. $X_1, \cdots, X_n$

$$\tau := \arg\max_t \{X_t\}$$

Can I leave the casino at time τ?

Def. A non-negative-integer-valued r.v. T is a stopping time if the event $\{T=n\}$ is determined by $(X_0, X_1, \cdots X_n)$ (on $\mathcal{F}_n$) for any $n = 0, 1, 2, \cdots$

eg. $(X_n)_{n\geq 0}$ SRW

$T := \inf\{t \geq 0 : X_t = 10\}$ is a stopping time.

eg $T = 5$

eg. $T := \inf\{t \geq 0, |X_t| = 10\}$

eg. $T := \inf\{n \geq 2, X_{n-2} = 10\}$

eg. ~~For MC,~~ $T_{ij}^{(k)}$ (k-th visit to i)

e.g. $T := \inf\{n \geq 0 : X_{n+1} = 10\}$ is not a stopping time.

e.g. T_1, T_2 are both stopping times,

so are $\min\{T_1, T_2\}$, $\max\{T_1, T_2\}$