STA447 / 2006 Lecture 11

Recall. Martingales. stopping times.

$\mathbb{E}[X_{t+1} | \mathcal{F}_t] = X_t$

At time T, you know the stopping time is reached.

e.g. (k-th) hitting time

Motivation: Play a game $\{X_n\}_{n \geq 1}$: amount of money

T : stopping time chosen by gambler

X_T : amount of money at the end.

Question $\mathbb{E}[X_T] \overset{?}{=} \mathbb{E}[X_0]$

(Counter-example. $\{X_n\}_{n \geq 1}$ SRW on $\mathbb{Z}$,
and $T := \inf\{n : X_n = 5\}$)

Lemma: If $\{X_n\}_{n \geq 0}$ is a martingale,
and T stopping time, and
$\boxed{\exists M < +\infty \text{ s.t. } T \leq M \ (\text{w.p. } 1)}$
Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Much stronger than $T < +\infty$ a.s.

e.g. Geometric r.v. $\mathbb{P}(T \geq n) = 2^{-n}$.

$$\mathbb{E}[X_T] - \mathbb{E}[X_0] = \mathbb{E}[X_T - X_0]$$

$$= \mathbb{E}\left[\sum_{k=1}^{T} (X_k - X_{k-1})\right]$$

$$= \mathbb{E}\left[\sum_{k=1}^{M} (X_k - X_{k-1}) \mathbb{1}_{k \le T}\right]$$

$$= \sum_{k=1}^{M} \mathbb{E}\left[(X_k - X_{k-1}) \mathbb{1}_{k \le T}\right] = 0.$$

Each term $= \mathbb{E}\left[(X_k - X_{k-1}) \cdot (1 - \mathbb{1}_{\{k-1 \ge T\}})\right]$

$$= \mathbb{E}\left[\mathbb{E}\left[(X_k - X_{k-1}) \,\middle|\, \mathcal{F}_{k-1}\right] \cdot (1 - \mathbb{1}_{\{k-1 \ge T\}})\right]$$

$\mathbb{E}[aX] = a \cdot \mathbb{E}[X]$

$= 0.$

$$\boxed{\begin{array}{l} \text{Cannot take } M = +\infty \\ \text{Require } \displaystyle\sum_{k=1}^{+\infty} |X_k - X_{k-1}| \cdot \mathbb{1}_{\{T \ge k\}} < +\infty \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(not useful)} \end{array}}$$

Thm (optional stopping)

$\{X_n\}_{n \ge 0}$ is martingale, and T stopping time, if $\mathbb{P}(T < +\infty) = 1$

(i) $\mathbb{E}[|X_T|] < +\infty.$

(ii) $\lim_{n \to +\infty} \mathbb{E}[X_n \mathbb{1}_{T > n}] = 0.$

(We know that $\mathbb{E}[\mathbb{1}_{T > n}] \to 0$)

Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Proof: $\forall m \in \mathbb{N}$, $S_m := \min(T, m)$. stopping time.

By Lemma,

$$(\forall m) \qquad \mathbb{E}[X_{S_m}] = \mathbb{E}[X_0].$$

$$X_{S_m} = X_T \cdot 1_{T \leq m} + X_m \cdot 1_{T > m}$$

$$X_T = X_T \, 1_{T \leq m} + X_T \cdot 1_{T > m}.$$

$$\text{Error} = \mathbb{E}[X_T \cdot 1_{T>m}] - \mathbb{E}[X_m \cdot 1_{T>m}].$$

Suffices to show $\mathbb{E}[X_T \cdot 1_{T>m}] \to 0$

$\mathbb{E}[X_m \cdot 1_{T>m}] \to 0$ Immediate by assumption.

$1_{T>m} \to 0$ (a.s.)

$\mathbb{E}[|X_T|] < +\infty$ So by DCT

$\mathbb{E}[X_T \cdot 1_{T>m}] \to 0.$

$$0 \leq \left| \mathbb{E}[X_T] - \underbrace{\mathbb{E}[X_{S_m}]}_{=\mathbb{E}[X_0] \ (\forall m)} \right| \leq |\text{Error}| \longrightarrow 0$$

So $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Corollary If $\{X_n\}_{n\geq 0}$ satisfies $\mathbb{P}(T<+\infty)=1$

$\exists M<+\infty$ $|X_n|\cdot\mathbb{1}_{n\leq T}\leq M$ a.s., then OST holds

Deterministic

(i) $|X_T|\leq M$ a.s. So $\mathbb{E}|X_T|<+\infty$

(ii) $\lim\limits_{n\to+\infty}\mathbb{E}\left[|X_n|\cdot\mathbb{1}_{T>n}\right]$

$\leq M\cdot\lim\limits_{n\to+\infty}\mathbb{E}\left[\mathbb{1}_{T>n}\right]=0$

(because $\mathbb{P}(T<+\infty)=1$)