$$\text{Grade} = \max \begin{cases} 25\% \times \text{midterm 1} + 25\% \times \text{midterm2} + 50\% \times \text{final} \\ 33\% \times \text{midterm 1} + 67\% \times \text{final} \\ 33\% \times \text{midterm 2} + 67\% \times \text{final} \\ 100\% \times \text{final.} \end{cases}$$

STA 447/2006 Lecture 12

Martingales. $(X_n)_{n\geq 1}$ $\cancel{\mathbb{E}[|X_n|] < +\infty}$

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$$

Stopping time: Know when to stop

$\{T=n\}$ is determined by $\{X_1, X_2, \ldots, X_n\}$

OST. Under some conditions $\begin{cases} T \text{ being bounded} \\ \boxed{\lim_{n\to+\infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{T>n}] = 0} \;(*) \end{cases}$.

$\mathbb{E}[X_T] = \mathbb{E}[X_0]$

(X is MG, T stopping time).

eg. Gambler's ruin.

$X_0 = a$

$$X_{t+1} = \begin{cases} X_t + 1 & \text{w.p. } 1/2 \\ X_t - 1 & \text{w.p. } 1/2. \end{cases}$$

$(X_n)_{n\geq 0}$

Martingale.

$$T := \inf\{n > 0 : X_n = 0 \text{ or } X_n = c\}.$$

$$a = \cancel{\mathbb{E}[X_0]} \overset{?}{=} \mathbb{E}[X_T] = c \cdot \mathbb{P}(X_T = c) + \cancel{0 \cdot \mathbb{P}(X_T = 0)}$$

$$\mathbb{P}(\text{win}) = \mathbb{P}(X_T = c) = \frac{a}{c}.$$

Justify "?".

$$\mathbb{E}[|X_n| \mathbb{1}_{T>n}]$$
$$\le \mathbb{E}[c \cdot \mathbb{1}_{T>n}] = c \cdot \mathbb{P}(T > n) \to 0$$

because $\mathbb{P}(T < +\infty) = 1$

Suppose
$$X_{t+1} = \begin{cases} X_t + 1 & \text{w.p. } p \\ X_t - 1 & \text{w.p. } 1-p \end{cases}$$

$$Y_n = \left(\frac{1-p}{p}\right)^{X_n} \quad \text{for } n = 0, 1, 2, \cdots$$

$$\mathbb{E}[Y_{n+1} | \mathcal{F}_n] = p \cdot \left(\frac{1-p}{p}\right)^{X_n+1} + (1-p) \cdot \left(\frac{1-p}{p}\right)^{X_n-1}$$

$$= \left(\frac{1-p}{p}\right)^{X_n} \cdot \{(1-p) + p\}$$

$$= Y_n.$$

$T := \inf\{n > 0 : X_n = 0 \text{ or } X_n = c\}$.

$$\left(\frac{1-p}{p}\right)^a = \mathbb{E}[Y_0] \overset{?}{=} \mathbb{E}[Y_T] = \mathbb{P}(X_T = c)\cdot\left(\frac{1-p}{p}\right)^c + \mathbb{P}(X_T = 0)\cdot 1$$

$$\mathbb{P}(X_T = c) = \frac{\left(\frac{1-p}{p}\right)^a - 1}{\left(\frac{1-p}{p}\right)^c - 1}.$$

Justify "?": when $n < T$

$$|Y_n| \le \max\left(\left(\frac{1-p}{p}\right)^c, 1\right).$$

Apply same arguments.

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e.g. Gambler's ruin, $p = 1/2$.

$\mathbb{E}[T]$?

Define $M_n = X_n^2 - n$.

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \mathbb{E}[X_{n+1}^2 - n - 1 \mid X_n] = M_n.$$

$$X_{n+1} = X_n + \varepsilon_n$$

$$X_{n+1}^2 = X_n^2 + 1 + 2\underbrace{\varepsilon_n X_n}_{\mathbb{E}[\cdots \mid X_n] = 0.}$$

$$a^2 = \mathbb{E}[M_0] \overset{?}{=} \mathbb{E}[M_T] = \mathbb{E}[X_T^2] - \mathbb{E}[T]$$

$$\mathbb{E}[X_T^2] = c^2 \cdot \mathbb{P}(X_T = c) + 0 = a \cdot c$$

$$\mathbb{E}[T] = a \cdot (c-a).$$

Justify "?"!

$$\mathbb{E}\left[|M_n| \cdot \mathbb{1}_{\{T>n\}}\right] \le (c^2+n) \cdot \mathbb{P}(T>n) \longrightarrow 0.$$

From MC theory, we know

$$\mathbb{P}(T>n) \le C \cdot \rho^n \quad (\text{for some } \rho < 1).$$

$\le \mathbb{P}($not achieving c- consecutive wins in n rounds$) \le$