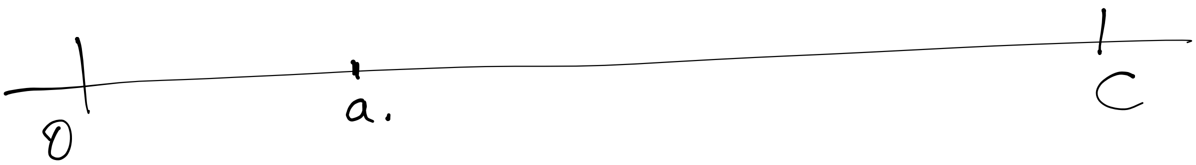
STA 447/2006 Lecture 13.

Recall OST. $\mathbb{E}\left[|X_n| \mathbb{1}_{\{T>n\}}\right] \to 0$ as $n \to +\infty$.

e.g. Gambler's ruin. $T = \inf\{t : X_t = 0 \text{ or } c\}$

$$\mathbb{P}(T > n) \le c \rho^n. \quad \text{for some } \rho < 1$$




$\forall a$ — $\mathbb{P}_a(T > c) \le 1 - p^c$

$$\mathbb{P}_a(T > n) \le (1 - p^c)^{n/c}$$

Generalizable to finite state space MC

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$$\mathbb{E}\left[|X_n| \mathbb{1}_{\{T>n\}}\right]$$

$|X_n|$: Integrable $\forall n$.

$\mathbb{1}_{\{T>n\}}$: Prob $\to 0$.

For fixed r.v. $X$;

$$\mathbb{E}\left[|X| \mathbb{1}_A\right] = \mathbb{E}\left[|X| \cdot \mathbb{1}_{A \cap \{|X|>K\}}\right] + \mathbb{E}\left[|X| \cdot \mathbb{1}_{A \cap \{|X| \le K\}}\right].$$

$$\le \mathbb{E}\left[|X| \cdot \mathbb{1}_{\{|X|>K\}}\right] + K \cdot \mathbb{P}(A).$$

From integrability. $\lim\limits_{K\to+\infty} \mathbb{E}\left[|X|\mathbb{1}_{\{|X|>K\}}\right]=0.$

Fact. $\forall \varepsilon>0, \exists K$ s.t. $\mathbb{E}\left[|X|\mathbb{1}_{|X|>K}\right] \leq \varepsilon.$

For MG. Uniform integrability.

$\forall \varepsilon>0, \exists K$ s.t. $\mathbb{E}\left[|X_n|\mathbb{1}_{|X_n|>K}\right] \leq \varepsilon. \quad (\forall n\in\mathbb{N}).$

Thm. $(X_n)_{n\geq 0}$ u.i., $T$ satisfies $T<+\infty$ a.s. and $\mathbb{E}[|X_T|]<+\infty.$

Then $\mathbb{E}[X_T]=\mathbb{E}[X_0].$

$\forall\varepsilon \exists K$

Proof. $\mathbb{E}\left[|X_n|\mathbb{1}_{A_n}\right]=\mathbb{E}\left[|X_n|\cdot\mathbb{1}_{A_n\cap\{|X_n|>K\}}\right]+\mathbb{E}\left[|X_n|\cdot\mathbb{1}_{A_n\cap\{|X_n|\leq K\}}\right]$

$(A_n=\{T>n\})$, $\leq \underbrace{\mathbb{E}\left[|X_n|\cdot\mathbb{1}_{\{|X_n|>K\}}\right]}_{\leq\varepsilon.}+\underbrace{K\cdot\mathbb{P}(A_n)}_{\to 0.}$

$\forall\varepsilon>0. \quad \limsup\limits_{n\to+\infty} \mathbb{E}\left[|X_n|\cdot\mathbb{1}_{T>n}\right] \leq \varepsilon.$

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Fact. If $\exists C<+\infty$, s.t. $\mathbb{E}[X_n^2]\leq C \quad \forall n$, then u.i.

Proof: $\mathbb{E}\left[|X_n|\cdot\mathbb{1}_{|X_n|\geq K}\right] \leq \sqrt{\mathbb{E}[|X_n|^2]}\cdot\sqrt{\mathbb{P}(|X_n|\geq K)}.$

$$\mathbb{P}(|X_n| \geq K) \leq \frac{\mathbb{E}[X_n^2]}{K^2}.$$

So $\mathbb{E}\left[|X_n| \cdot \mathbb{1}_{|X_n| \geq K}\right] \leq \frac{\mathbb{E}[X_n^2]}{K} \leq \frac{C}{K}.$

e.g. $X_n = \sum_{j=1}^{n} \frac{1}{j} \cdot Z_j$ where $Z_j$ iid. $\begin{cases} 1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$

$$\mathbb{E}[X_n^2] \leq \underbrace{\sum_{j=1}^{+\infty} \frac{1}{j^2}}_{= \pi^2/6.} < +\infty$$

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Playing w/ truncation arguments —— Wald's theorem.

$X_n = \sum_{j=1}^{n} Z_j$ where $Z_j$'s iid, $\mathbb{E}[|Z_j|] < +\infty$,

$\mathbb{E}[Z_j] = m.$

$\{X_n - n \cdot m\}_{n \geq 0}$ is MG.

Thm (Wald). If $\mathbb{E}[T] < +\infty$, then $\mathbb{E}[X_T] = m \cdot \mathbb{E}[T]$.

By optional stopping Lemma

$$\mathbb{E}\left[X_{n \wedge T} - m \cdot (n \wedge T)\right] = 0 \quad (\forall n).$$

$(a \wedge b = \min(a,b))$.

$$\lim_{n\to+\infty} \mathbb{E}[n \wedge T] = \mathbb{E}[T] \ (<+\infty).$$

$$\left| \mathbb{E}[X_{n\wedge T}] - \mathbb{E}[X_T] \right|$$

$$\leq \mathbb{E}\left[ \sum_{m=n+1}^{T} |Z_m| \cdot \mathbb{1}_{\{T>n\}} \right].$$

$$= \mathbb{E}\left[ \sum_{m=n+1}^{+\infty} |Z_m| \cdot \mathbb{1}_{\{T\geq m\}} \right]$$

$$= \sum_{m=n+1}^{+\infty} \mathbb{E}\left[ |Z_m| \cdot \mathbb{1}_{\{T\geq m\}} \right].$$

Key observation: $\{T \leq m-1\} = \bigcup_{j=1}^{m-1} \{T = j\}$.

$\{T \geq m\}$ is determined by $Z_1, Z_2, \cdots Z_{m-1}$,

and therefore independent with $Z_m$.

$$\mathbb{E}\left[ |Z_m| \cdot \mathbb{1}_{\{T\geq m\}} \right] = \mathbb{E}[|Z_m|] \cdot \mathbb{P}(T \geq m).$$

So $\displaystyle\sum_{m=n+1}^{+\infty} \mathbb{E}\left[ |Z_m| \cdot \mathbb{1}_{\{T\geq m\}} \right] \leq \mathbb{E}[|Z_1|] \cdot \sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m).$

$$\sum_{m=1}^{+\infty} \mathbb{P}(T \geq m) = \mathbb{E}[T] < +\infty$$

So $\sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m) \to 0$ as $n \to +\infty$.

$$\left| \mathbb{E}[X_{n \wedge T}] - \mathbb{E}[X_T] \right| \to 0 \quad \text{as } n \to +\infty.$$

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Martingale convergence

Idea: $\mathbb{P}(M_n \to M_\infty) = 1$ ?

eg. Gambler's ruin. $(X_{n \wedge T})_{n \geq 0}$

$(X_n)_{n \geq 0}$ SRW, $T$ = hitting time of $\{0, c\}$

eg. $\sum_{j=1}^{+\infty} Z_j / j$.

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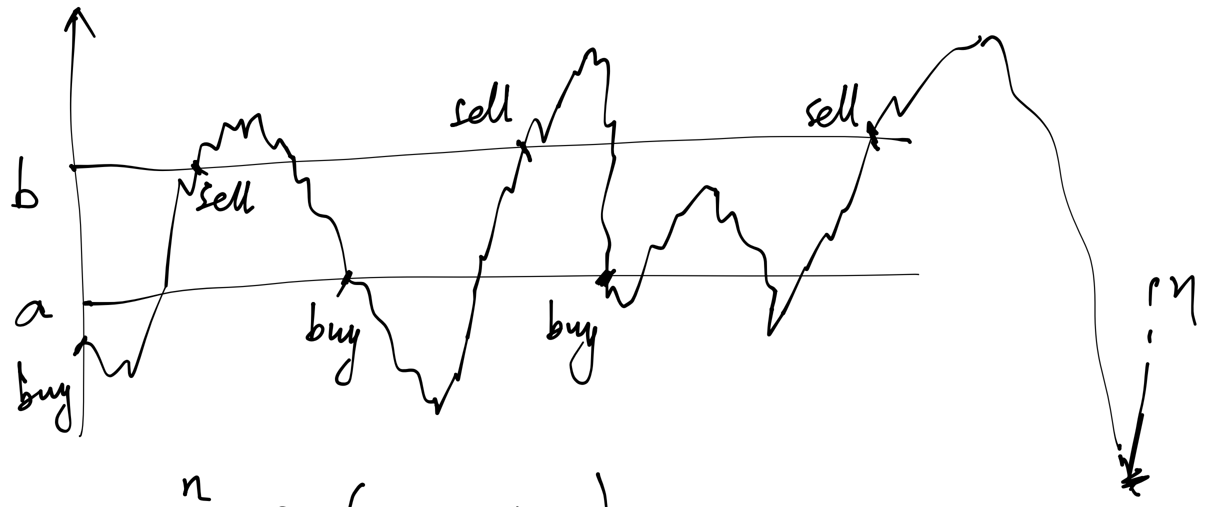
Thm. If $\mathbb{E}[|M_n|] \leq C < +\infty$ $(\forall n)$

Then $M_n \to M_\infty$ a.s.

Proof idea: "upcrossing"!




$$W_n = \sum_{j=1}^{n} B_j (M_j - M_{j-1})$$

where $B_j = \begin{cases} 1 & \text{when } B_{j-1}=1,\ X_{j-1}<b \\ & \text{or } B_{j-1}=0,\ X_{j-1} \le a. \\ 0 & (\text{otherwise}). \end{cases}$

$$\mathbb{E}[W_{n+1} \mid \mathcal{F}_n] = W_n + \mathbb{E}[B_{n+1}(M_{n+1} - M_n) \mid \mathcal{F}_n] = W_n.$$

Let $U_n :=$ # upcrossings up to time $n$.

$$W_n \ge (b-a) \cdot U_n - |M_n - a|.$$

$$\mathbb{E}[W_0] = \mathbb{E}[W_n] \ge (b-a) \cdot \mathbb{E}[U_n] - \mathbb{E}[|M_n - a|]$$

$\forall n.$ $\quad \mathbb{E}[U_n] \le \frac{1}{b-a} \left( \mathbb{E}[|M_n|] + a \right) \le \frac{a+C}{b-a}.$

# upcrossing for the entire process $< +\infty$ (a.s.) $\forall a, b$.

$\forall a, b.$ $\mathbb{P}(\#\text{ upcrossing } [a,b] < +\infty) = 1.$

Want $\mathbb{P}(\forall a, b, \ \#\text{ upcrossing } [a,b] < +\infty) = 1.$

$$\mathbb{P}\left(\exists a, b \in \mathbb{Q}, \ \#\text{ upcrossing } [a,b] = +\infty\right)$$

$$\leq \sum_{a,b \in \mathbb{Q}} \mathbb{P}\left(\#\text{ upcrossing } [a,b] = +\infty\right) = 0$$

Implies $\limsup_{n \to +\infty} M_n = \liminf_{n \to +\infty} M_n$ a.s.

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Remark $\mathbb{E}[|M_n|] \leq C$ can be replaced by

- $M_n \geq c$ for some $c \in \mathbb{R}$.

$$\left( W_n \geq (b-a) U_n - |M_n - a| \mathbb{1}_{\{M_n \leq a\}} \right).$$

- $M_n \leq c$ for some $c \in \mathbb{R}$.