STA 447/2006 Lecture 14.

Under one of $\begin{cases} \mathbb{E}[|X_n|] \leq C < +\infty & \forall n. \\ X_n \geq C & \text{for some } C \\ X_n \leq C & \text{for some } C. \end{cases}$

then $X_n \xrightarrow{a.s.} X_\infty$.

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e.g. SRW. $(X_n)_{n\geq 0}$

$T := \inf\{t \geq 0 : X_t = -1\}$

$Y_n = X_{n\wedge T}$ $\qquad Y_n \geq -1$. a.s.

$Y_n \xrightarrow{a.s.} Y_\infty$ $\qquad \mathbb{P}(T < +\infty) = 1$.

$Y_\infty = -1$ (a.s.)

$-1 = \mathbb{E}[Y_\infty] \neq \mathbb{E}[Y_0] = 0$.

Fact If $(X_n)_{n\geq 0}$ MG, and u.i., $\mathbb{E}[|X_\infty|] < +\infty$.

then $\mathbb{E}[X_\infty] = \mathbb{E}[X_0]$.

Recall. u.i. means $\forall \varepsilon > 0$, $\exists K$

s.t $\mathbb{E}\left[|X_n| \mathbb{1}_{|X_n| > K}\right] \leq \varepsilon$ $\quad (\forall n \in \mathbb{N})$. (*)

$\mathbb{E}[|X_n|] \leq \mathbb{E}\left[|X_n| \mathbb{1}_{|X_n|>K_1}\right] + \mathbb{E}\left[|X_n| \mathbb{1}_{|X_n| \leq K_1}\right] \leq K_1 + 1$.

Proof of the fact: $\forall \varepsilon > 0$, $\exists K_\varepsilon$ s.t. $(*)$ holds.

$$\mathbb{E}[X_n] = \underbrace{\mathbb{E}[X_n \mathbb{1}\{|X_n| \le K_\varepsilon\}]}_{\substack{(\text{DCT}) \ \downarrow \ n \to +\infty \\ \mathbb{E}[X_\infty \mathbb{1}\{|X_\infty| \le K_\varepsilon\}]}} + \underbrace{\mathbb{E}[X_n \mathbb{1}\{|X_n| > K_\varepsilon\}]}_{\le \varepsilon}$$

$$\left|\mathbb{E}[X_n] - \mathbb{E}[X_\infty]\right| \le \left|\mathbb{E}[X_n \mathbb{1}\{|X_n| \le K_\varepsilon\}] - \mathbb{E}[X_\infty \mathbb{1}\{|X_\infty| \le K_\varepsilon\}]\right| + \mathbb{E}[|X_n| \mathbb{1}\{|X_n| \ge K_\varepsilon\}] + \mathbb{E}[|X_\infty| \mathbb{1}\{|X_\infty| \ge K_\varepsilon\}]$$

$\exists K'_\varepsilon$ s.t. $\mathbb{E}[|X_\infty| \mathbb{1}\{|X_\infty| \ge K'_\varepsilon\}] \le \varepsilon$.

Take $K = \max\{K_\varepsilon, K'_\varepsilon\}$, $n \to +\infty$

$$\limsup_{n \to +\infty} \left|\mathbb{E}[X_n] - \mathbb{E}[X_\infty]\right| \le 2\varepsilon.$$

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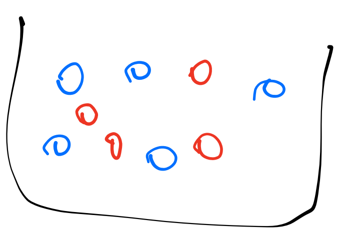
e.g. $M_n = \sum_{j=1}^{n} \frac{1}{j} \cdot X_j$ where $X_j \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2} \end{cases}$

$M_n \xrightarrow{a.s.} M_\infty$.

$0 = \mathbb{E}[M_n] = \mathbb{E}[M_\infty]$

e.g. Polya's urn.



$M_n :=$ Proportion of blue balls.

For each time, put a new ball w.p. $=$ proportion of color

$$M_n \xrightarrow{a.s.} M_\infty. \qquad \mathbb{E}[M_n] = \mathbb{E}[M_\infty].$$

e.g. (X_n) irreducible MC

Function f harmonic if

[Intuition: Discrete analogue of $\Delta f(x) = 0 \quad \forall x$].

$$f(x) = \sum_{y \in S} p(x,y) f(y).$$

When integrable $(f(X_n))_{n \geq 0}$ is MG.

T = hitting time of $z \in S$.

$$M_n := f(X_{n \wedge T}) \text{ is MG.}$$

Fact If f is harmonic & bdd, and P recurrent.

then f is constant.

Proof: $\mathbb{P}_x(T < +\infty) = 1$.

$(M_n)_{n \geq 0}$ uniformly bdd MG.

$$M_n \xrightarrow{a.s.} M_\infty \qquad \mathbb{E}[M_n] = \mathbb{E}[M_\infty] = f(z).$$

$$\begin{aligned} &\qquad\qquad \| \\ f(x) &= \mathbb{E}[M_0] \end{aligned}$$

In the transient case. Fix $z \in S$

$$f(x) = \begin{cases} \mathbb{P}_x(T_z < +\infty) \ (= f_{xz}), & (x \neq z) \\ 1 & (x = z). \end{cases}$$

From f-expansion, f is harmonic.

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e.g. $X_1, X_2, \dots$ i.i.d

$$\mathbb{P}\left(X_i = \tfrac{3}{2}\right) = \mathbb{P}\left(X_i = \tfrac{1}{2}\right) = \tfrac{1}{2}.$$

$M_0 = 1$, $M_n = X_1 \cdot X_2 \cdot \dots \cdot X_n$.

$(M_n)_{n \geq 0}$ is MG. $\mathbb{E}[M_n] = \mathbb{E}[M_0] = 1$.

$$M_n \xrightarrow{a.s.} M_\infty.$$

$$\mathbb{E}[\log(X_i)] < 0.$$

$$\log(M_n) = \sum_{i=1}^{n} \log(X_i)$$

By SLLN, $\log(M_n) \xrightarrow{a.s.} -\infty.$

So $M_\infty = 0.$