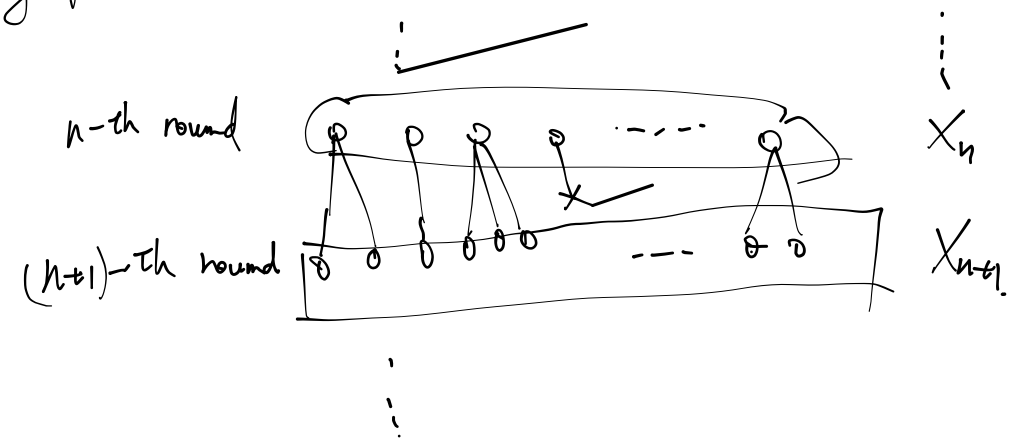
STA447/2006 Lecture 15

MG convergence $\begin{cases} \mathbb{E}[|X_n|] \le C < +\infty. & \text{uniformly bdd from above/below} \\ \text{u.i.} \quad \mathbb{E}[|X_\infty|] < +\infty. & \& \quad \mathbb{E}[X_\infty] = \mathbb{E}[X_0] \\ & \mathbb{E}[|X_\infty - X_n|] \longrightarrow 0. \\ \text{(not covered) For } p>1 & \mathbb{E}[|X_n|^p] \le C < +\infty \\ \text{(Implies u.i.)} & \mathbb{E}[|X_n - X_\infty|^p] \longrightarrow 0. \end{cases}$

Branching process.

X_0



μ := "offspring distribution" on $\{0, 1, 2, \cdots\}$.

$$X_{n+1} = Z_{n,1} + Z_{n,2} + \cdots + Z_{n,X_n}$$

where $Z_{n,i}$'s are iid r.v. from μ.

$Z_{n,i}$:= # children of the i-th individual in n-th round.

Question: die out or not?

$$\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \sum_{i=1}^{X_n} \mathbb{E}[Z_{n,i} | \mathcal{F}_n] = X_n \cdot \mathbb{E}_\mu[Z].$$

$m := \mathbb{E}_\mu[Z] = \cancel{\sum_{i\geq 0} i\cdot\mu_i}$

$Y_n = m^{-n} X_n$ is a MG.

eg. If $m<1$, $\mathbb{E}[X_n] = m^n \cdot \mathbb{E}[X_0] \longrightarrow 0$ (as $n \to +\infty$)

So $X_n \xrightarrow{P} 0$.

eg. If $m>1$, but $\mu_0 > 0$.

w.p. >0 die out

$\mathbb{E}[X_n] \to +\infty$.

$\mathbb{P}(X_n \to +\infty) > 0$.

How about $m=1$? (Assume $\mu_1 < 1$).

X_n is a MG. $X_n \geq 0$ $(\forall n)$.

$X_n \xrightarrow{a.s.} X_\infty$

X_n integer $\Longrightarrow$ $\exists\, T < +\infty$. a.s.

$X_n = X_\infty$ after $n \geq T$.

So $X_n \to 0$ a.s.

Doob's martingale $\quad X \quad \mathbb{E}[|X|] < +\infty.$

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_n \subseteq \cdots\cdots$$

$$M_n := \mathbb{E}[X \mid \mathcal{F}_n]$$

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n\big].$$

$$= \mathbb{E}[X \mid \mathcal{F}_n] = M_n.$$

Uniform integrability. $\forall \varepsilon, \exists K$ s.t.

$$\mathbb{E}\big[|M_n| \cdot \mathbb{1}_{\{|M_n| \geq K\}}\big] \leq \mathbb{E}\big[|X| \cdot \mathbb{1}_{\{|X| \geq K\}}\big] \leq \varepsilon.$$

$$X_n \xrightarrow[L^1]{a.s.} X_\infty$$

Indeed, if $\mathcal{F}_\infty = \bigcup_{n=1}^{+\infty} \mathcal{F}_n \qquad X_\infty = \mathbb{E}[X \mid \mathcal{F}_\infty].$

e.g. Posterior consistency.

$\theta \sim \pi$ (prior distribution)

$$X_1, X_2, \cdots X_n \mid \theta \overset{iid}{\sim} P_\theta$$

Posterior distribution $\quad \Pi(\theta \mid X_1, X_2 \cdots X_n) \propto \pi(\theta) \cdot p_\theta(X_1) \cdots p_\theta(X_n)$

Goal: estimate $g(\theta)$

$$\hat{g}_n := \mathbb{E}[g(\theta) \mid X_1, X_2, \cdots, X_n].$$

$$\hat{g}_n \overset{?}{\to} g(\theta).$$

$\mathcal{F}_n = (X_1, X_2, \cdots, X_n)$ $\quad (\hat{g}_n)_{n \geq 0}$ is Doob MG.

$$\hat{g}_n \xrightarrow[L^1]{a.s.} g_\infty = \mathbb{E}[g(\theta) \mid \mathcal{F}_\infty]$$

g_∞ is a r.v. determined by $\mathcal{F}_\infty = (X_i)_{i=1}^{+\infty}$.

e.g. when $\exists (\tilde{g}_n)_{n \geq 1}$ s.t. $\tilde{g}_n \xrightarrow{a.s.} g(\theta)$. $(\forall \theta)$.

$g(\theta)$ can be determined by $\mathcal{F}_\infty$.

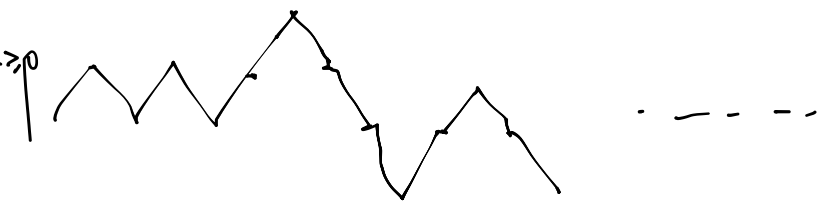
$$\mathbb{E}[g(\theta) \mid \mathcal{F}_\infty] = g(\theta).$$

So $\hat{g}_n \xrightarrow{a.s.} g(\theta)$

Brownian motion.

Motivation: SRW in 1-d.

$(X_n)_{n\geq 0}$



$\varepsilon_i \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$

At time n. $X_n = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$

$$\frac{1}{\sqrt{n}} X_n \xrightarrow{d} N(0,1)$$

For finite k fixed. $0 < t_1 < t_2 < \cdots t_k < +\infty$.

$$\left(X_{\lfloor nt_1 \rfloor}, X_{\lfloor nt_2 \rfloor}, \cdots X_{\lfloor nt_k \rfloor} \right)$$

$$\sqrt{n}. \begin{bmatrix} X_{\lfloor nt_1 \rfloor} \\ X_{\lfloor nt_2 \rfloor} - X_{\lfloor nt_1 \rfloor} \\ X_{\lfloor nt_3 \rfloor} - X_{\lfloor nt_2 \rfloor} \\ \vdots \\ X_{\lfloor nt_k \rfloor} - X_{\lfloor nt_{k-1} \rfloor} \end{bmatrix} \xrightarrow{d} N\left(0, \begin{bmatrix} t_1 & & & 0 \\ & t_2 - t_1 & & \\ & & \ddots & \\ 0 & & & t_k - t_{k-1} \end{bmatrix}\right).$$

$$\left(\frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor} \right)_{0 \leq t \leq T} \xrightarrow{d} \text{something.}$$

Defn. $(B_t)_{t\geq 0}$ is BM iff.

1. $B_0 = 0$
2. Normally distributed $B_t \sim N(0,t)$.
3. Independent normal increments.

   For $t>s$, $B_t - B_s \sim N(0, t-s)$
   independent of $(B_t)_{0\leq t\leq s}$.

4. (implied by 3).

$$\mathrm{cov}(B_t, B_s) = \min(s,t).$$

$0 \leq s < t$

$$\mathbb{E}[B_t B_s] = \underbrace{\mathbb{E}[(B_t - B_s)\cdot B_s]}_{=0} + \underbrace{\mathbb{E}[B_s^2]}_{s}.$$

5. mapping $t \mapsto B_t$ is continuous a.s.

Intuition: at time $[t, t+\Delta t]$, make increment $\sim N(0, \Delta t)$.

$$\lim_{\Delta t\to 0} \frac{B_{t+\Delta t} - B_t}{\Delta t} \qquad B_{t+\Delta t} - B_t \sim N(0, \Delta t)$$

$$|B_{t+\Delta t} - B_t| \simeq \sqrt{\Delta t}$$

Limit does not exist

Easy to prove: $\forall t$, $\mathbb{P}(B \text{ is non-diff at } t) = 1$.

Fact. $\mathbb{P}(\forall t \in (0, \infty),\ B \text{ is non-diff at } t) = 1$.

Fact: BM is Markov.

$(B_t - B_s)_{t \ge s}$ is a BM, indp of the past

"Strong Markov property".

stopping time in cts-time case:

Def T is a stopping time if

the event $\{T \le t\}$ is determined by

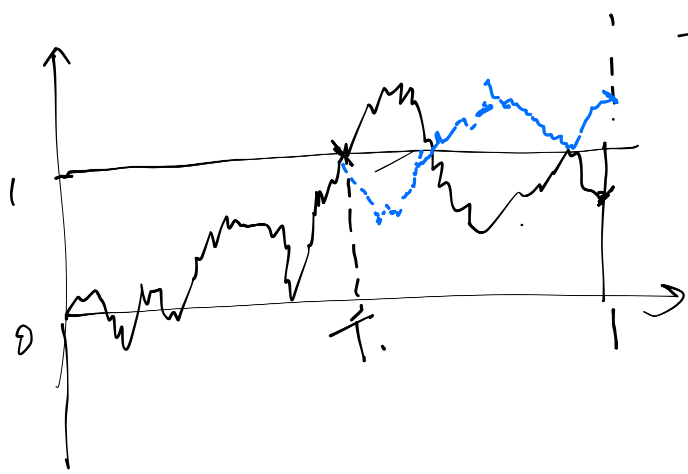
$\mathcal{F}_t$, i.e. $(B_s)_{0 \le s \le t}$.

e.g. Hitting time of $a \in \mathbb{R}$.

Fact: T is a stopping time. $\mathbb{P}(T < +\infty) = 1$

then $(B_{t+T} - B_T)_{t \ge 0}$ is a BM

independent of $(B_s)_{0 \le s \le T}$.



$T := \inf\{t : B_t = 1\}$

$\mathbb{P}(T \le 1)$?

"Reflection principle".

At time $t=1$ $\begin{cases} B_t \leq 1 \\ B_t > 1. \quad \text{(already hit)} \end{cases}$

$$\underbrace{\mathbb{P}(B_1 \geq 1)}_{\text{Computable.}} = \underbrace{\mathbb{P}(T \leq 1)}_{\text{Goal}} \cdot \underbrace{\mathbb{P}(B_1 \geq 1 \mid T \leq 1)}_{\text{By strong Markov.}} \cancel{+ \mathbb{P}(T > 1) \cdot 0.}$$

By strong Markov.

$(B_t - 1)_{t \geq T}$ is BM.

Conditionally on $(B_t)_{0 \leq t \leq T}$,

$$\mathbb{P}\left(B_1 - B_T \geq 0 \mid (B_t)_{0 \leq t \leq T},\ T \leq 1\right)$$

$$= \mathbb{P}\left(\mathcal{N}(0, 1-t) \geq 0\right) = \frac{1}{2}.$$

Conclusion:

$$\mathbb{P}(T \leq 1) = 2 \cdot \mathbb{P}(X_1 \geq 1) = 2 \cdot \int_1^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

$$\mathbb{P}(T_a \leq t) = 2 \cdot \mathbb{P}(X_t \geq a) = \underline{2 \int_a^{+\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, dx}$$

$$\mathbb{P}(T_a \leq t) = \mathbb{P}\left(\max_{0 \leq s \leq t} B_s \geq a\right).$$