STA 447/2006 Lecture 16

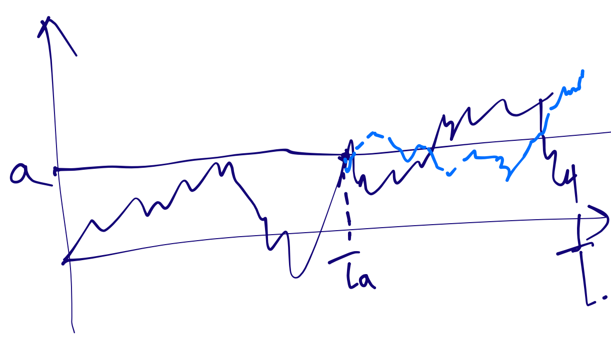
Correction. Doob's MG. $M_n = \mathbb{E}[X \mid \mathcal{F}_n]$.

$$\mathbb{E}\left[|M_n| \mathbb{1}_{|M_n|>K}\right] \leq \mathbb{E}\left[|X| \cdot \mathbb{1}_{\max |M_i| > K}\right].$$

Require "maximal inequality".

Able to bound $\max_{1\le i\le n} |M_i|$.

Recall: Strong Markov property.



$$\mathbb{P}(T_a \leq T) = 2\cdot \mathbb{P}(X_T \geq a)$$

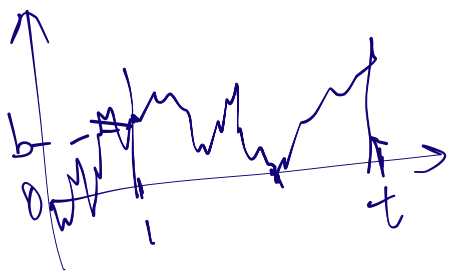
$$\parallel$$

$$\mathbb{P}\left(\max_{0\le t\le T} B_t \geq a\right)$$

$(a\geq 0)$.

Conclusion: $\max_{0\le t\le T} B_t \overset{d}{=} |B_T|$.

e.g. $\mathbb{P}\left(X_s = 0 \text{ for some } s \in [1,t] \,\middle|\, X_1 = b \right)$.



$= \mathbb{P}_0\left(T_{-b} \leq t-1 \right)$.

$= 2 \cdot \mathbb{P}_0\left(B_{t-1} \leq -b \right)$.

Integrate over b.

Following (hard) calculus exerci:

$$\mathbb{P}\left(X_s = 0 \text{ for some } s \in [1,t] \right)$$

$$= 1 - \frac{2}{\pi} \cdot \arctan\left(\frac{1}{\sqrt{t-1}} \right).$$

Fact: $(B_t)_{t \geq 0}$ is a martingale

$\mathbb{E}\left[|B_t| \right] < \infty \quad (\forall t)$

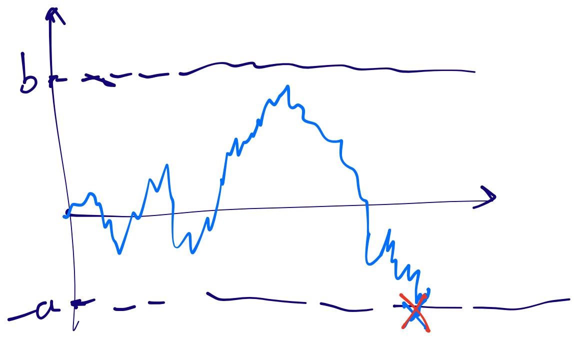
and $\mathbb{E}\left[B_t \,\middle|\, \mathcal{F}_s \right] = B_s$.

$0 \leq s \leq t$

Can use OST, MG convergence under some conditions.

e.g. $T = \inf\{t \geq 0 : B_t = -a \text{ or } B_t = b\}$.

$\mathbb{P}(B_T = -a)$?



$(B_t)_{t \geq 0}$ is a MG,

$|B_t \cdot \mathbb{1}_{t \leq T}| \leq \max(a, b)$

Apply "bounded up to T" version of OST.

$$\mathbb{E}[B_T] = \mathbb{E}[B_0] = 0.$$

$$\| $$

$$-a \cdot \mathbb{P}(B_T = -a) + b \cdot \mathbb{P}(B_T = b).$$

$$\mathbb{P}(B_T = -a) + \mathbb{P}(B_T = b) = 1$$

So we have $\mathbb{P}(B_T = -a) = \dfrac{b}{a+b}$.

$\limsup |B_t| \to +\infty$ as $t \to +\infty$ a.s.

How about $\mathbb{E}[T]$?

$Y_t := B_t^2 - t$

For $0 \le s \le t$

$$\mathbb{E}[B_t^2 \mid \mathcal{F}_s] = \mathbb{E}\left[B_s^2 + (B_t - B_s)^2 + 2 \cdot B_s (B_t - B_s) \,\middle|\, \mathcal{F}_s\right].$$

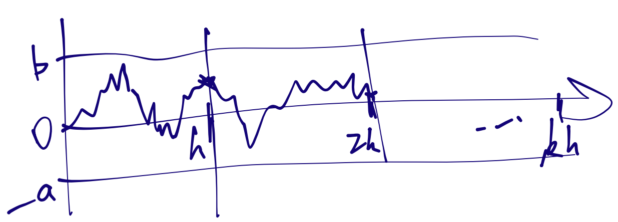
$$= B_s^2 + (t - s) + 0$$

By OST.

$$0 = \cancel{\mathbb{E}}[B_T^2 - T] = -\mathbb{E}[T] + \mathbb{P}(B_T = -a) \cdot (-a)^2 + \mathbb{P}(B_T = b) \cdot b^2$$

$$\mathbb{E}[T] = a \cdot b.$$

Verify tail properties of T: similar to SRW.



$\forall$ starting x, w.p. $p > 0$, hit $\{-a, b\}$ within time h, Bernoulli trials.

"Zero sets of BM".

Facts: (Scaling properties).

$(B_t : t \geq 0)$ is a BM.

(i). $\forall a > 0$, $\frac{1}{\sqrt{a}} \cdot B_{at}$ is also a BM.

(ii) $(t \cdot B_{1/t})_{t \geq 0}$ is also a BM.

(Easy to verify).

At time $t \to +\infty$, $(B_t)_{t \geq 0}$ hits 0 infinitely often.

$\forall \varepsilon > 0$. $(B_t)_{0 \leq t \leq \varepsilon}$ visits 0 infinitely often.

$\overset{d}{\|}$

$(s \cdot B_{1/s})_{0 \leq s \leq \varepsilon}$.

$$Z := \{ t \in \mathbb{R}^+, B_t = 0 \}.$$