STA 447/2006 Lecture 17.

Integration & differentiation w.r.t. BM.

Goal: to make sense of $\int_0^t Y_s \, dB_s$

Y_s "nice".

B_s : gambling game

Y_s : the gambling strategy with $[s, s+\Delta s]$.

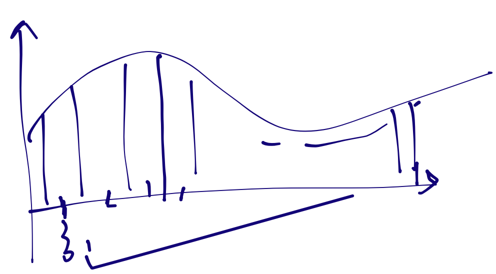
Require Y:

— Y_t is determined by $(B_s)_{0 \le s \le t}$ (i.e. $\mathcal{F}_t$).

— $\mathbb{E}[|Y_t|^2] < +\infty$ $(\forall t)$ — $\int_0^t \mathbb{E}[Y_s^2] \, ds < +\infty$.

Recall : Riemann integral.

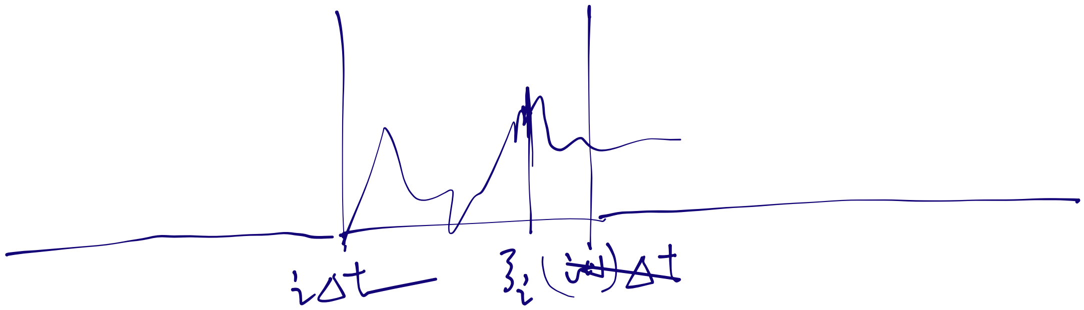
$\int_a^b f(t) \, dt$



$= \lim_{\Delta t \to 0} \sum_{i=0}^{\frac{b-a}{\Delta t}} f(\xi_i) \cdot \Delta t.$

Want to define

$$\int_0^T Y_s \, dB_s \overset{?}{\neq} \lim_{\Delta t \to 0} \sum_{i=0}^{T/\Delta t} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot Y_{\xi_i}.$$



e.g. $$\int_0^t B_s \, dB_s \overset{?}{\neq} \lim_{\Delta t \to 0} \sum_{i=0}^{t/\Delta t} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{(i+\frac{1}{2})\Delta t}$$

$$\sum_{i=0}^{t/\Delta t} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{(i+\frac{1}{2})\Delta t} = \boxed{\sum_{i=0}^{t/\Delta t} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot \left(B_{(i+\frac{1}{2})\Delta t} - B_{i\Delta t}\right)}$$

$$+ \underbrace{\sum_{i=0}^{t/\Delta t} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{i\Delta t}.}_{\text{zero - mean.}}$$

$\longrightarrow 0$ in Riemann Integral.

For Stochastic integration.

each term i.i.d. $\mathbb{E}[\text{----}] = \frac{\Delta t}{2}.$

sum $\xrightarrow{\text{LLN}} \frac{t}{2}.$

Criterion for choosing the midpoint:

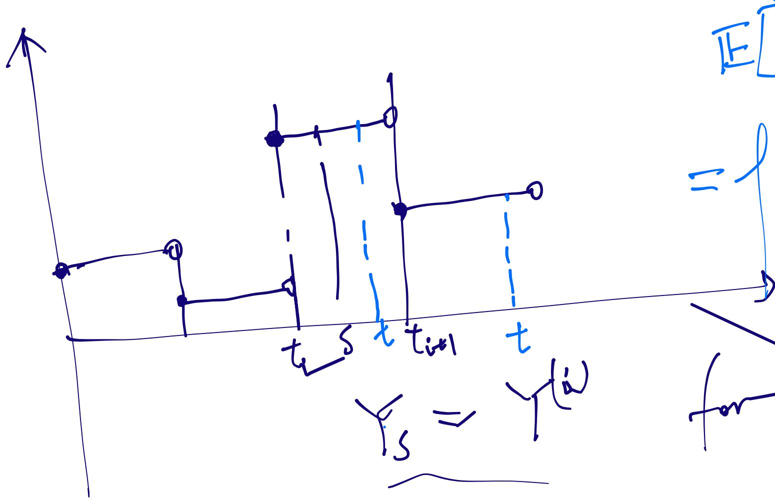
make $\left(\int_0^t Y_s \, dB_s\right)_{t \geq 0}$ a martingale

Need to choose $\xi_i = i \cdot \Delta t$ (left endpt).

Roadmap toward a rigorous defn:

— Stoch int for a piecewise const Y_s

— Take the limit.



$s < t$

$\mathbb{E}[Z_t \mid \mathcal{F}_s] = \begin{cases} Z_s & \text{if } s, t \in [t_i, t_{i+1}] \\ Z_s & \text{if not.} \end{cases}$

$Y_s = Y^{(i)}$ for $s \in [t_i, t_{i+1})$.

$$Z_t = \int_0^t Y_s \, dB_s \overset{\text{dfn}}{=} \sum_{i=0}^{N} Y^{(i)} \cdot \left(B_{t_{i+1}} - B_{t_i}\right).$$

Easy to verify $\left\{ \begin{array}{l} \text{martingale} \\ \int_0^t (aY_s + bY_s')\,dB_s = a\int_0^t Y_s\,dB_s + b\int_0^t Y_s'\,dB_s. \\ \text{(Itô isometry)} \quad \mathbb{E}[Z_t^2] = \int_0^t \mathbb{E}[Y_s^2]\,ds. \quad (*). \end{array} \right.$

Verify (*).

$$\int_0^t \mathbb{E}[Y_s^2]\,ds = \sum_i \mathbb{E}\left[(Y^{(i)})^2\right](t_{i+1}-t_i).$$

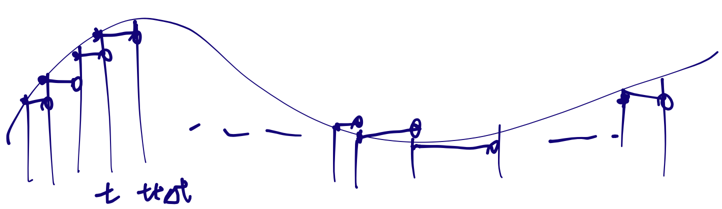
$$Z_t^2 = \underbrace{\sum_i (Y^{(i)})^2 \cdot (B_{t_{i+1}} - B_{t_i})^2}_{\mathbb{E}[\cdot] = \sum_i \mathbb{E}[(Y^{(i)})^2]\cdot(t_{i+1}-t_i).} + \text{cross terms.}$$

"cross term" for $i<k$

$$0 = \mathbb{E}\Big[\underbrace{Y^{(i)}\cdot Y^{(k)}\cdot (B_{t_{i+1}} - B_{t_i})}_{\text{determined by } \mathcal{F}_{t_k}.} \cdot \underbrace{(B_{t_{k+1}} - B_{t_k})}_{\text{indp of } \mathcal{F}_{t_k}.}\Big]$$

$(Y_s : s \ge 0)$ cts, adapted to $(\mathcal{F}_s)_{s\ge 0}$.

$$\left(\tilde{Y}^{(n)}_s\right)_{s\ge 0}.$$



$$\tilde{Z}_t^{(N)} := \int_0^t \tilde{Y}_s^{(N)} \, dB_s.$$

$$\tilde{Z}_t^{(n)} - \tilde{Z}_t^{(m)} = \int_0^t \left(\tilde{Y}_s^{(n)} - \tilde{Y}_s^{(m)} \right) dB_s$$

$$\mathbb{E}\left[\left| \tilde{Z}_t^{(n)} - \tilde{Z}_t^{(m)} \right|^2 \right] = \int_0^t \mathbb{E}\left| \tilde{Y}_s^{(n)} - \tilde{Y}_s^{(m)} \right|^2 ds. \to 0.$$

if $m, n \to +\infty$.

$$\tilde{Z}_t^{(n)} \xrightarrow{L^2} Z_t.$$

Try to verify Z_t satisfies MG, linearity, Itô isometry

"Notation": $F = \int f \, ds \iff dF = f \, ds.$

$$Z_t = \int_0^t X_s \, ds + \int_0^t Y_s \, dB_s \quad (\forall t).$$

$$\Updownarrow$$

$$dZ_t = X_t \, dt + Y_t \, dB_t$$

Itô's formula.

Motivation: Newton-Leibniz $\int_a^b f'(t)\,dt = f(b) - f(a)$.

Not true for stochastic integral

$$\underbrace{\int_0^t B_s\,dB_s}_{\text{mean } 0} \overset{?}{=} \underbrace{\frac{1}{2}\left(B_s^2 - B_0^2\right)}_{\text{mean } \frac{t}{2}}.$$

$$f(B_{\Delta t}) - f(B_0) \qquad (f \in C^2)$$

$$= f'(B_0)\cdot B_{\Delta t} + \frac{1}{2} f''(B_0)\cdot \underbrace{B_{\Delta t}^2}_{\text{of order } \Delta t.} + o\left(|B_{\Delta t}|^2\right)$$

$$f(B_t) - f(B_0)$$

$$= \sum_{j=0}^{n-1} f'\left(B_{\frac{jt}{n}}\right)\cdot\left(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}\right) + \frac{1}{2}\sum_{j=0}^{n-1} f''\left(B_{\frac{jt}{n}}\right)\left(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}\right)^2$$

$$+ \sum_{j=0}^{n-1} o\left(\frac{1}{n}\right), = o(1).$$

$$\approx \int_0^t f'(B_s)\,dB_s + ?$$

In general. interested in

$$\lim_{n\to+\infty} \sum_{j=0}^{n-1} g\left(B_{\frac{jt}{n}}\right) \cdot \underbrace{\left(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}\right)^2}_{\overset{d}{=} \frac{t}{n} U^2}$$

where $U \sim N(0,1)$.

eg. $g \equiv 1$

$$\frac{1}{n} \sum_{j=0}^{n-1} \cdot n\left(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}\right)^2 \xrightarrow{SLLN} t.$$

Natural guess. $\int_0^t g(B_s)\, ds$.

Roadmap :- constant g

$\downarrow$

piecewise const. g

$\downarrow$

cts processes (includes cts funcs of BM).

Thm. $f \in C^2$

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\, dB_s + \frac{1}{2}\int_0^t f''(B_s)\, ds.$$

eg $f(x) = x^2$.

$$B_t^2 = \int_0^t 2B_s \, dB_s + t.$$

$$\int_0^t B_s \, dB_s = \underbrace{\frac{1}{2} B_t^2 - \frac{1}{2} t}.$$

MG (used for Gambler's ruin).

$$\left(f(B_t) - f(B_0) - \frac{1}{2} \int_0^t f''(B_s) \, ds \right)_{t \geq 0} \text{ is MG.}$$

eg. $X_t = e^{B_t}$

$$X_t - 1 = \int_0^t e^{B_s} \, dB_s + \frac{1}{2} \int_0^t e^{B_s} \, ds.$$

$$dX_t = X_t \, dB_t + \frac{1}{2} X_t \, dt.$$

Extensions. $\quad dZ_t = X_t \, dt + Y_t \, dB_t$

$$df(Z_t) = f'(Z_t) \, dZ_t + \frac{1}{2} f''(Z_t) \cdot Y_t^2 \, dt$$

$$= f'(Z_t) X_t \, dt + f'(Z_t) Y_t \, dB_t + \frac{1}{2} f''(Z_t) Y_t^2 \, dt$$

$$f(Z_t) - f(Z_0) = \sum_{j=0}^{n-1} f'\left(Z_{\frac{jt}{n}}\right) \underbrace{\left(Z_{\frac{(j+1)t}{n}} - Z_{\frac{jt}{n}}\right)}$$

$$+ \frac{1}{2} \sum_{j=0}^{n-1} f''\left(Z_{\frac{jt}{n}}\right) \cdot \left(Z_{\frac{(j+1)t}{n}} - Z_{\frac{jt}{n}}\right)^2 .$$

$$+ o(1).$$

$$Z_{\frac{(j+1)t}{n}} - Z_{\frac{jt}{n}} \approx X_{\frac{jt}{n}} \cdot \left(\frac{t}{n}\right) + Y_{\frac{jt}{n}} \cdot \left(B_{\frac{(j+1)t}{n}} - B_{\frac{jt}{n}}\right).$$