STA 447/2006 Lecture 18

$$dZ_t = X_t\,dt + Y_t\,dB_t$$

Then $$df(Z_t) = f'(Z_t)\,dZ_t + \underbrace{\frac{1}{2} Y_t^2 f''(Z_t)\,dt}$$

$$\frac{1}{2}\int_0^t Y_s^2 f''(Z_s)\,ds = \frac{1}{2}\lim_{n\to+\infty}\sum_{j=0}^{n-1}\left(Z_{\frac{(j+1)t}{n}} - Z_{\frac{jt}{n}}\right)^2 f''\left(Z_{\frac{jt}{n}}\right).$$

$$\langle Z\rangle_t = \lim_{n\to+\infty}\sum_{j=0}^{n-1}\left(Z_{\frac{(j+1)t}{n}} - Z_{\frac{jt}{n}}\right)^2.$$ "Quadratic variation"

$t \mapsto \langle Z\rangle_t$ cts, non dec, differentiable

$$\int_0^t Y_s^2 f''(Z_s)\,ds = \int_0^t f''(Z_s)\,d\langle Z\rangle_s$$

$$df(Z_t) = f'(Z_t)\,dZ_t + \frac{1}{2}f''(Z_t)\,d\langle Z\rangle_t.$$

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e.g. $$d(Z_t^2) = 2Z_t\,dZ_t + d\langle Z\rangle_t.$$

$$d\left[\left(\frac{Z_t^{(1)} + Z_t^{(2)}}{2}\right)^2\right] = \ldots\ldots$$

$$d\left[\left(\frac{Z_t^{(1)} - Z_t^{(2)}}{2}\right)^2\right] = \ldots\ldots$$

Taking their difference $\Longrightarrow$ Product rule

$$d\left(Z_t^{(1)} Z_t^{(2)}\right) = Z_t^{(1)} dZ_t^{(2)} + Z_t^{(2)} dZ_t^{(1)} + d\langle Z^{(1)}, Z^{(2)}\rangle_t.$$

$$\langle Z^{(1)}, Z^{(2)}\rangle_t = \lim_{n\to\infty} \sum_{j=0}^{n-1} \left(Z^{(1)}_{\frac{(j+1)t}{n}} - Z^{(1)}_{\frac{jt}{n}}\right)\cdot\left(Z^{(2)}_{\frac{(j+1)t}{n}} - Z^{(2)}_{\frac{jt}{n}}\right).$$

If $\quad dZ_t^{(i)} = X_t^{(i)} dt + Y_t^{(i)} dB_t \quad$ for $i = 1, 2$.

$$\langle Z^{(1)}, Z^{(2)}\rangle_t = \int_0^t Y_s^{(1)} Y_s^{(2)} ds.$$

$$df(t, Z_t) = \partial_t f(t, Z_t) dt + \partial_x f(t, Z_t)\, dZ_t + \frac{1}{2}\partial_x^2 f(t, Z_t)\cdot d\langle Z\rangle_t.$$

$\boxed{\partial_t f,\ \partial_x f,\ \partial_x^2 f \text{ exists, and } \cancel{\text{cts}}}$

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eg. $\quad f(t,x) = e^{at+bx}, \qquad Z_t = e^{at+bB_t}$

$$dZ_t = a\cdot e^{at+bB_t} dt + b\cdot e^{at+bB_t} dB_t + \frac{b^2}{2} e^{at+bB_t} dt.$$

i.e. $$dZ_t = \left(a+\frac{b^2}{2}\right)Z_t\,dt + b\,Z_t\,dB_t.$$

For SDE $$dZ_t = rZ_t\,dt + b\,Z_t\,dB_t$$

Solution: $$Z_t = \exp\left(bB_t + \left(r-\frac{b^2}{2}\right)t\right).$$

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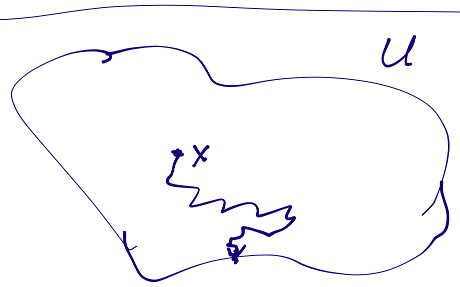
e.g. $p_t(y) :=$ density of d-dim BM at y

(starting ~~from x~~)

$$\frac{\partial p_t}{\partial t}(y) = \frac{1}{2}\Delta_y p_t(y)\ldots$$

$$= \frac{1}{2}\sum_{j=1}^{d}\frac{\partial^2}{\partial y_j^2}p_t(y).$$



bdd, connected region U.

$$\tau := \inf\{t>0 : X_t \in \partial U\}.$$

Define function g on ∂U.

$$f(x) = \mathbb{E}_x\left[g(B_\tau)\right].$$

$$f(B_t) = f(x) + \int_0^t \langle \nabla f(B_t), dB_t \rangle \quad \text{Martingale}$$
$$\cancel{+ \frac{1}{2}\int_0^t \Delta f(B_t)\, dt.}$$

Choose f s.t. $\begin{cases} \Delta f = 0 & \text{in } U \\ f = g & \text{on } \partial U. \end{cases}$

Apply OST: $f(x) = \mathbb{E}_x[f(B_\tau)]$.

Why useful. $U := \{x \in \mathbb{R}^d : R_1 \le \|x\|_2 \le R_2\}$

$g = 1$ on $\{x : \|x\|_2 = R_2\}$

$g = 0$ on $\{x : \|x\|_2 = R_1\}$

$f = \mathbb{P}(\text{BM hits outer ball first}),$

$$f(x) = \begin{cases} \dfrac{\log \|x\|_2 - \log R_1}{\log R_2 - \log R_1} & (d=2) \\[2ex] \dfrac{R_1^{2-d} - \|x\|_2^{2-d}}{R_1^{2-d} - R_2^{2-d}} & (d \ge 3) \end{cases}$$

$$\lim_{R_2 \to +\infty} \mathbb{P}_x(\text{hit } R_2 \text{ before } R_1) = \begin{cases} 0 & (d=2) \\ >0 & (d \geq 3). \end{cases}$$