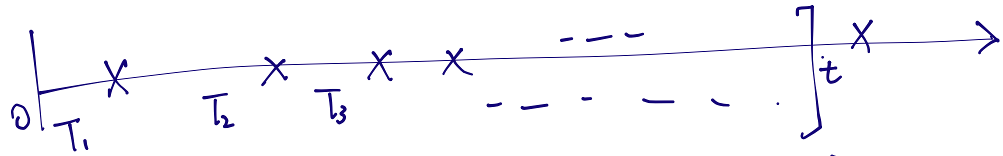
STA 447/2006 Lecture 19.

Goal: $T_i \overset{iid}{\sim} p$



$N(t) :=$ # marked pts within $[0,t]$.

Special case: $p_\lambda(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$ $\quad \mathbb{P}(T > t) = e^{-\lambda t}$

Exponential distr with rate λ.

$T \sim p_\lambda \quad \mathbb{E}[T] = \frac{1}{\lambda}, \quad \mathrm{var}(T) = \frac{1}{\lambda^2}$

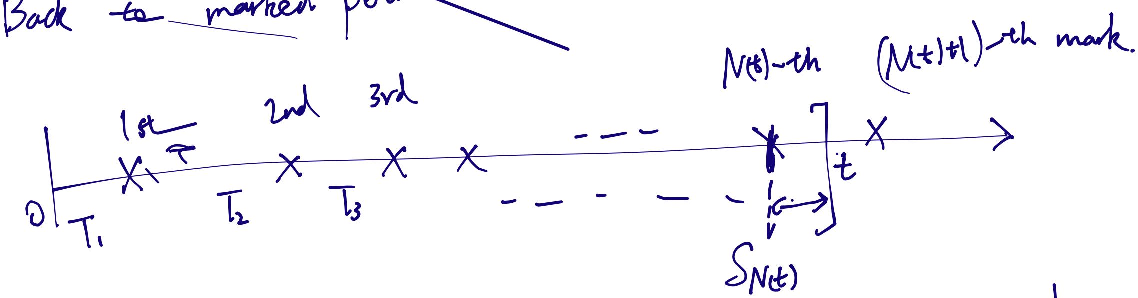
If $T \sim \mathrm{Exp}(1)$, then $\frac{T}{\lambda} \sim \mathrm{Exp}(\lambda)$.

"Lack of memory": $\forall\, t, s > 0$, $T \sim \mathrm{Exp}(\lambda)$

$$\mathbb{P}(T > t+s \mid T > t) = \frac{\mathbb{P}(T > t+s)}{\mathbb{P}(T > t)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}}$$

$$= e^{-\lambda s} = \mathbb{P}(T \geq s).$$

Back to marked points



$$\mathbb{P}\left(\begin{matrix}\text{Next mark}\\ \text{takes time} > s\end{matrix} \,\middle|\, \mathcal{F}_t\right) = \mathbb{P}\left(T_{N(t)+1} > (t - S_{N(t)}) + s \,\middle|\, \underbrace{T_{N(t)+1} > t - S_{N(t)}}\right)$$

$$= \mathbb{P}(T > s).$$

Next mark has not arrived yet at time t.

"Poisson process with intensity λ"

$N(t) :=$ # marks in $[0,t]$

If $T_i \overset{iid}{\sim} Exp(\lambda)$.

Detour: Poisson distribution.

Call $N \sim Poi(\lambda)$. if

$$\mathbb{P}(N=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!} \quad \text{for } n = 0, 1, 2, \ldots$$

Properties: $\mathbb{E}[N] = \lambda$, $var(N) = \lambda$

If $X_i \sim Poi(\lambda_i)$ $i=1,2,\cdots n$ independent

Then $X_1 + X_2 + \cdots + X_n \sim Poi(\lambda_1 + \lambda_2 + \cdots + \lambda_n)$

$X \sim Binom(n, p)$ where $p = \lambda/n$.

$$\mathbb{P}(X=k) = \frac{n!}{(n-k)!\,k!} \cdot \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{n\cdot(n-1)\cdots(n-k+1)}{n^k} \cdot \frac{1}{k!}\lambda^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$(n \to +\infty)$ $\quad 1 \qquad e^{-\lambda}$

$$= e^{-\lambda} \cdot \lambda^k / k!$$

Prop: $N(t) \sim Poi(\lambda t)$ $\forall t$.

Proof: $\mathbb{P}(N(t) = n) = \mathbb{P}(S_n \leq t < S_{n+1})$

$$= \int_0^t f_{S_n}(s) \cdot \mathbb{P}(T_{n+1} > t-s)\, ds$$

where f_{S_n} is the density of S_n.

$$S_n = \sum_{i=1}^n T_i,$$

Lemma. $T_1, T_2, \ldots, T_n \overset{iid}{\sim} Exp(\lambda)$

and $S_n = \sum_{i=1}^{n} T_i$, then

$$f_{S_n}(t) = \lambda \cdot e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \quad (\forall t).$$

(Proof: induction)

Taking Lemma as given . . .

$$\mathbb{P}(N(t)=n) = \int_0^t \lambda \cdot e^{-\lambda s} \cdot \frac{(\lambda s)^{n-1}}{(n-1)!} \cdot e^{-\lambda(t-s)} \, ds$$

$$= e^{-\lambda t} \cdot \frac{(\lambda t)^n}{n!}$$

Fact: $(N(t))_{t \geq 0}$ has independent Poisson increments.

$t_0 < t_1 < \cdots < t_n$ then

$N(t_1) - N(t_0), N(t_2) - N(t_1), \ldots, N(t_n) - N(t_{n-1})$ indp

each following $Poi(\lambda(t_i - t_{i-1}))$.

Thm: Poisson Process $\Longleftrightarrow$ indp $Poi(\lambda t)$ increment.