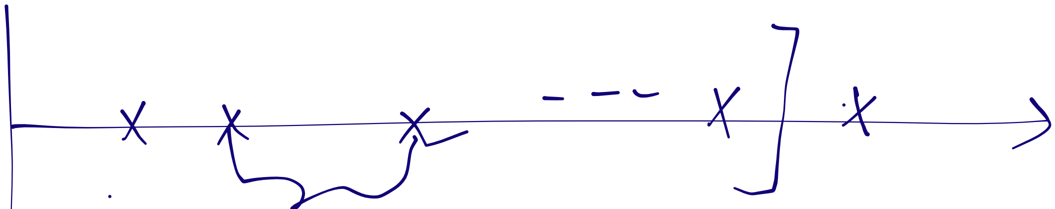
STA 447/2006 Lecture 20.

Recall. two ways of defining PP

(1)



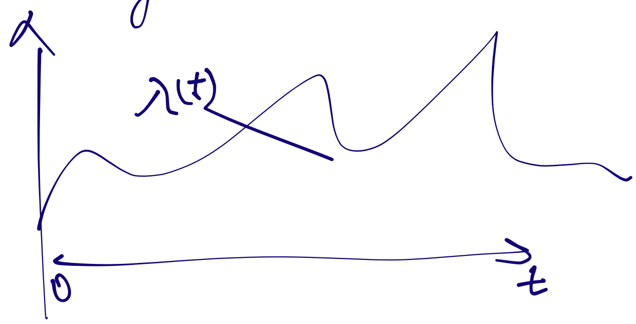
$\mathrm{Exp}(\lambda)$, independent.

$N(t) :=$ # marked pts up to time t.

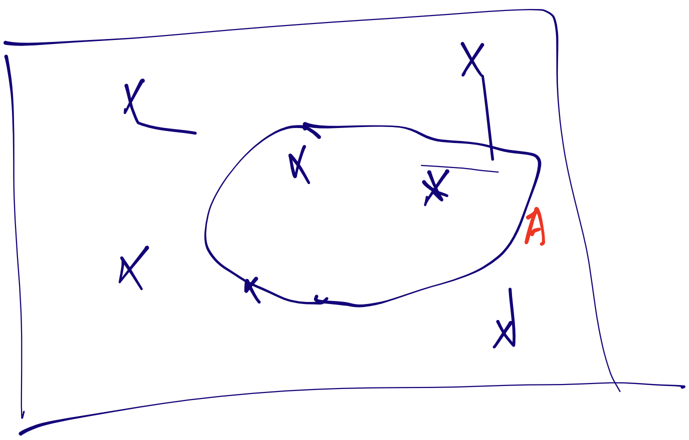
(2) $N(t) \sim \mathrm{Poi}(\lambda t)$

$N(t) - N(s) \sim \mathrm{Poi}(\lambda(t-s))$ independent of the past

Time - inhomogeneous PP.



$$N(t) - N(s) \sim \mathrm{Poi}\left(\int_s^t \lambda(x)\,dx\right).$$



$N(A) :=$ # marked pts in A.

$N(A) \sim \mathrm{Poi}\left(\int_A \lambda(x)\,dx\right)$.

and for disjoint A, B

$N(A)$ independent of $N(B)$.

Alternative characterization.

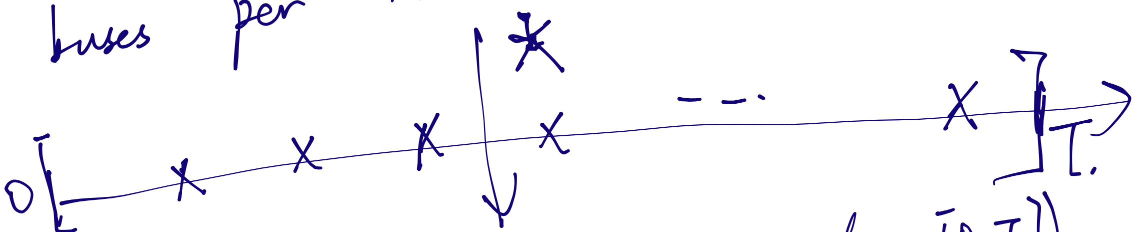
Prop. As $h \to 0$, (i) $\mathbb{P}\big(N(t+h) - N(t) = 1\big) = \lambda h + o(h)$.

(integer-valued) (ii) $\mathbb{P}\big(N(t+h) - N(t) \geq 2\big) = o(h)$.

Fact: Any stochastic process w/ indep increments satisfying (i)(ii) is PP w/ rate λ.

e.g. Bus waiting time "paradox"

λ buses per hour



Arrive at random time (uniform $[0,T]$).

Wait time $\sim$ Uniform$\left[0, \frac{1}{\lambda}\right]$.

Average wait time $= \frac{1}{2\lambda}$.

$T_i \overset{iid}{\sim} \mathrm{Unif}\left(\left[0, \frac{n}{\lambda}\right]\right)$ $i = 1, 2, \dots, n$

Fix a, b, $n \to \infty$

\# marked pts in $[a,b] \sim \mathrm{Binom}\left(n, \frac{\lambda(b-a)}{n}\right)$

$\longrightarrow \mathrm{Poi}\big(\lambda(b-a)\big)$

Regardless of the customer's arrival time

$$\mathbb{E}[\text{waiting time}] = \mathbb{E}[T] \quad \text{where } T \sim \text{Exp}(\lambda)$$
$$= \frac{1}{\lambda}.$$

"Superposition ~~property~~":

$(N_1(t))_{t\geq 0}, (N_2(t))_{t\geq 0}$ indp PP. w/ intensity λ_1, λ_2

then $(N_1(t) + N_2(t))_{t\geq 0}$ is PP w/ intensity $\lambda_1+\lambda_2$

"Thining property"

$(N(t))_{t\geq 0}$ be PP w/ intensity λ

And each marked pt is independently of type i.
w.p. p_i for $i=1,2,\ldots$ $\left(\sum_i p_i = 1\right)$

$(N_i(t))_{t\geq 0}$ are PP w/ intensity $\underline{\lambda \cdot p_i}$
and independent.

Proof: (two types for simplicity)

If you have indp Poisson Process $\underline{N_1(t), N_2(t)}$

then

$$\mathbb{P}\left(N_1(t)=j,\ N_2(t)=k\right) = \frac{e^{-\lambda p_1 t}}{j!}(\lambda p_1 t)^j \cdot \frac{e^{-\lambda p_2 t}}{k!}(\lambda p_2 t)^k.$$

If you have $N(t) \sim \mathrm{Poi}(\lambda t)$

then draw $N_1(t) \sim \mathrm{Binom}(N(t), p_1)$.

$$N_2(t) = N(t) - N_1(t).$$

$$\mathbb{P}\left(N_1(t)=j,\ N_2(t)=k\right) = \frac{e^{-\lambda t}}{(j+k)!}(\lambda t)^{j+k} \cdot \binom{j+k}{j} \cdot p_1^j\, p_2^k$$