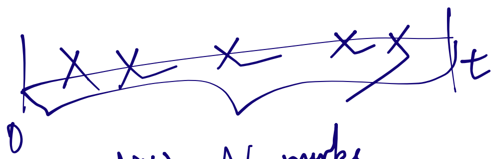
STA 447/2006 Lecture 21.

Suppose $(N(t))_{t\ge 0}$ PP w/ intensity $\lambda$.

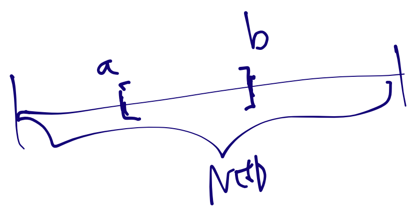
Fix $t$, $N(t) \sim \mathrm{Poi}(\lambda t)$.




$N(t) = N$ marks.

Claim: conditionally on $N(t) = N$,

marked pts $\overset{iid}{\sim}$ Unif $([0,t])$.




$\#$ marks in $[a,b] \sim \mathrm{Poi}(\lambda(b-a))$ $N_{in}(t)$

outside $[a,b] \sim \mathrm{Poi}(\lambda(t-b+a))$ (indp). $N_{out}(t)$

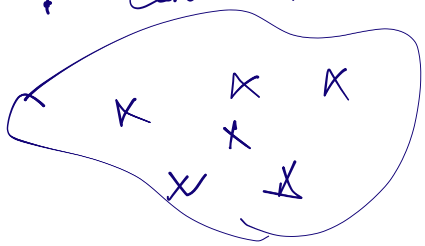
Conditioned on $N(t) = N_{in}(t) + N_{out}(t) = N$.

$$\mathbb{P}\left(N_{in}(t) = k \,\middle|\, N(t) = N\right) = \frac{N!}{k!\,(N-k)!} \cdot \left(\frac{b-a}{t}\right)^k \cdot \left(\frac{t-b+a}{t}\right)^{N-k}$$

Moreover, for PPP with intensity function $\lambda$,

For any set $A$ in the domain

- \# marked pts $\underline{N(A)} \sim \mathrm{Poi}\left(\int_A \lambda(x)\,dx\right)$
- Conditioned on $N(A)=N$, draw $N$ iid marked pts



from $\dfrac{\lambda}{\int_A \lambda(x)\,dx}$.

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\ Cts-time, discrete-space MC

Discrete state space $S$, init distribution $\nu$.

Defn (Markov process) $(X(t))_{t\geq 0}$ taking values in $S$,

$$\mathbb{P}(X_0 = i_0, X_{t_1} = i_1, X_{t_2} = i_2, \cdots X_{t_n} = i_n)$$

$$= \nu_{i_0} \cdot P^{(t_1)}_{i_0 i_1} \cdot P^{(t_2 - t_1)}_{i_1 i_2} \cdot \cdots \cdot P^{(t_n - t_{n-1})}_{i_{n-1} i_n}.$$

where $\left(P^{(t)}_{ij}\right)_{\substack{i,j \in S \\ t \geq 0}}$ transition probs.

$$P^{(0)}_{ij} = \begin{cases} 1 & (i=j) \\ 0 & (i \neq j). \end{cases}$$

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e.g. PP w/ intensity $\lambda$.

$$P^{(t)}_{ij} = \begin{cases} 0 & (j < i) \\ \dfrac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!} & (j \geq i). \end{cases}$$

In general, MC does ~~not imply~~

$$\lim_{t\to 0} P_{ij}^{(t)} = P_{ij}^{(0)} \quad (*)$$

"standard Markov process".

For this class, assume $(*)$ is true.

Kolmogorov–Chapman. (Discrete time: $P^{(n)} \cdot P^{(m)} = P^{(n+m)}$).

For $s,t \geq 0$, $P^{(s)} \cdot P^{(t)} = P^{(s+t)}$

Standard Markov $\Longrightarrow$ $P_{ij}^{(t)}$ is cts in $t$. $(\forall t)$.

Idea: Scalar case.

$$f(x+y) = f(x) \cdot f(y). \qquad f(x) = e^{ax}$$

$$a = \lim_{x\to 0} \frac{f(x) - f(0)}{x}$$

Defn (generator).

$$g_{ij} := \lim_{t\to 0} \frac{P_{ij}^{(t)} - P_{ij}^{(0)}}{t} \qquad (\text{for } i,j \in S).$$

(Roughly speaking, for small $t$

$$P^{(t)} \approx I + t \cdot G \qquad \text{where } G = (g_{ij})_{i,j\in S}).$$

Basic properties of $G$

- $g_{ii} = \lim_{t \to 0} \dfrac{p_{ii}^{(t)} - 1}{t} \le 0.$

- $g_{ij} = \lim_{t \to 0} \dfrac{p_{ij}^{(t)}}{t} \ge 0. \quad (i \ne j)$

- $\sum_{j \in S} g_{ij} = \lim_{t \to 0} \dfrac{\sum_{j \in S} \left( p_{ij}^{(t)} - p_{ij}^{(0)} \right)}{t} = 0.$

So $(-g_{ii}) = \sum_{\substack{j \in S \\ j \ne i}} g_{ij}$

e.g. PP w/ intensity $\lambda$.

$$g_{ij} = \lim_{t \to 0} \frac{p_{ij}^{(t)} - p_{ij}^{(0)}}{t} = \begin{cases} 0 & (j < i) \\ 0 & (j \ge i+2) \\ \lambda & (j = i+1) \\ -\lambda & (j = i) \end{cases}$$

$$\begin{bmatrix} -\lambda & \lambda & 0 & \cdots & & 0 \\ 0 & -\lambda & \lambda & 0 & \cdots & \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ \vdots & & & \ddots & \ddots & 0 \\ & 0 & & & \ddots & \ddots \end{bmatrix}$$

Compute transition probs using generator.

Want $P^{(t)} = \exp(t \cdot G)$

$$= I + tG + \frac{t^2 G^2}{2!} + \frac{t^3 G^3}{3!} + \cdots$$

Thm: If $G$ is generator, then $P^{(t)} = \exp(tG)$.

Proof idea:

$$P^{(t)} = \left(P^{(t/n)}\right)^n = \lim_{n\to+\infty} \left(P^{(t/n)}\right)^n.$$

$$\left(P^{(t/n)} = I + \frac{t \cdot G}{n} + o\left(\frac{1}{n}\right)\right) \quad = \lim_{n\to+\infty} \left(I + \frac{t}{n} G\right)^n$$

$$= \exp(tG).$$

Compute matrix exp.

$G = P \Lambda P^{-1}$ (work w/ diagonalizable case).

$$\exp(tG) = \sum_{n=0}^{+\infty} \frac{1}{n!} (tG)^n$$

$$= P\left(\sum_{n=0}^{+\infty} t^n \Lambda^n\right) P^{-1}$$

$(\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots))$

$$= P \cdot \mathrm{diag}(e^{t\lambda_1}, e^{t\lambda_2}, \ldots) \cdot P^{-1}.$$

Another way to compute.

$$\frac{dP^{(t)}}{dt} = G \cdot P^{(t)}$$

$S \times S$ -dimensional (Linear) ODE

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· Defn. $(\pi_i)_{i \in S}$ stationary distribution if $\pi G = 0$.

( equivalent, $\pi P^{(t)} = \pi$, $\forall t \geq 0$).

$$\sum_{i \in S} \pi_i g_{ij} = 0 \quad (\forall j \in S).$$

Defn. MC reversible w.r.t. $(\pi_i)_{i \in S}$ if

$$\pi_i g_{ij} = \pi_j g_{ji}.$$

(Reversible $\Rightarrow$ stationary : $\sum_{i \in S} \pi_i g_{ij} = \sum_{i \in S} \pi_j g_{ji} = 0.$)

Thm Irreducible MC, w/ stationary $\pi$.

($\forall i,j \in S$ $\exists t$ w.p. $>0$, go from $i$ to $j$ in time $t$).

$$\lim_{t \to +\infty} P_{ij}^{(t)} = \pi_j \quad (\forall i,j \in S).$$

Proof idea: Fixed $h > 0$. $P^{(h)}$ prob transition matrix.

$(X_{kh})_{k=0,1,\ldots}$. $\pi = P^{(h)} \pi$.

So $\lim_{k \to +\infty} P_{ij}^{(hk)} = \pi_j$ [convergence of DTMC].

Holding time $\forall h \geq 0$

So $\lim_{t\to\infty} P_{ij}^{(t)} = \pi_j$. $(\forall i,j \in S)$.

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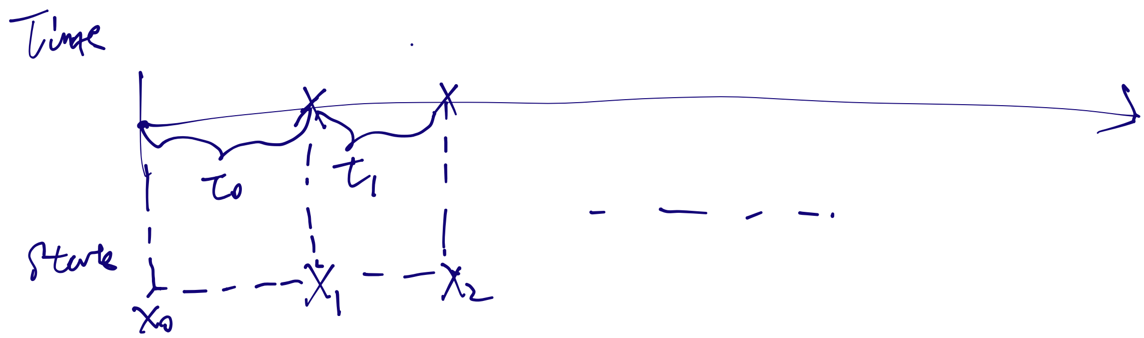
Construction of cts-time MCs.

Given generator $(g_{ij})_{i,j\in S}$.




$$\text{Time} \begin{cases} t_i \sim \mathrm{Exp}(-g_{ii}) & \text{If } g_{ii} > 0 \\ \text{Absorbing state} & \text{If } g_{ii} = 0. \end{cases}$$

Next-step transition:

$$\tilde{P}_{ij} := \begin{cases} \dfrac{g_{ij}}{-g_{ii}} & (\text{for } j \neq i) \\ 0 & (\text{otherwise}). \end{cases}$$

Transition follows $\tilde{P}$.

Claim: cts-time MC w/ generator $G$

$\stackrel{d}{=}$ process constructed above.

"Proof idea": for $t \to 0$, $P^{(t)} \approx I + tG$.

w.p. $\begin{cases} 1+tg_{i,i} & \text{stay at } i, \\ -tg_{i,i} & \text{move to } j \neq i \\ & \text{w.p. } t\cdot g_{i,j} \end{cases}$

\# trials till success $\sim \text{Geom}(1+tg_{i,i})$.

rescaling, $\longrightarrow \text{Exp}(-g_{i,i})$.

$\overbrace{|\,|\,|\,|\,|\,|\,|\,|\,|\ \cdots\ |}^{\text{Geom}(1+tg_{i,i})} \longrightarrow \text{Exp}(-g_{i,i})$.

$t$ for each