STA 447 Lecture 3.

$f_{ij} := P_i(\text{hitting } j)$.

f-expansion

$$f_{ij} = P_{ij} + \sum_{\substack{k \in S \\ k \neq j}} P_{ik} f_{kj}$$

eg. Gambler's ruin.

Initial money $a \in \mathbb{N}_+$

each time $\begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$

Game stops whenever one of two cond. happens.

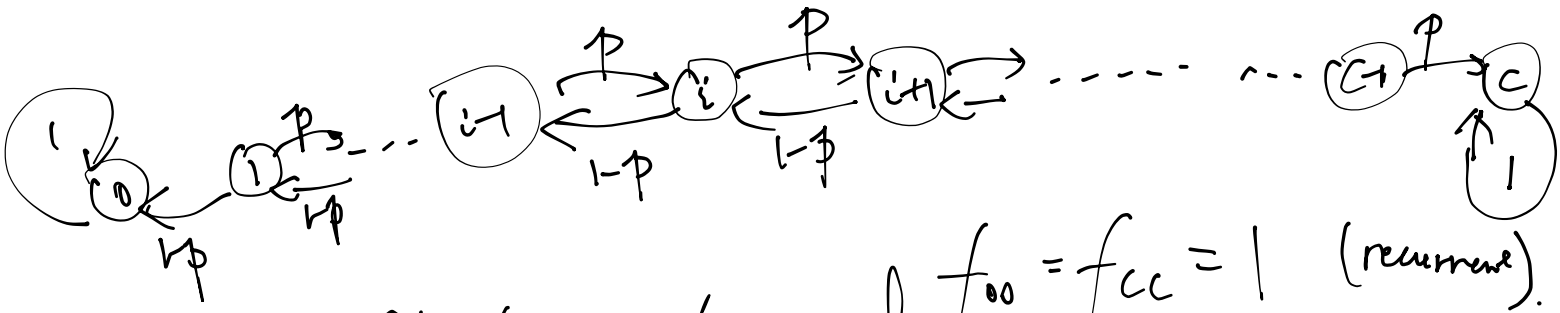
— Lose all money

— Amount of money reaches c.

$S = \{0, 1, 2, \cdots, c\}$

$v_a = 1, \quad v_i = 0$ for $i \neq a$

Transition:



Classification of states. $\begin{cases} f_{00} = f_{CC} = 1 & \text{(recurrent)} \\ f_{il} < 1 & (i \notin \{0, C\}) \text{ (transient)} \end{cases}$

$$f_{i0} = p_{i0} + \sum_{\substack{k \in S \\ k \neq 0}} p_{ik} f_{k0}$$

$$= \begin{cases} 1-p + p \cdot f_{20} & (i=1) \\ (1-p) \cdot f_{(i-1)0} + p \cdot f_{(i+1)0} & (i \geq 2) \end{cases}$$

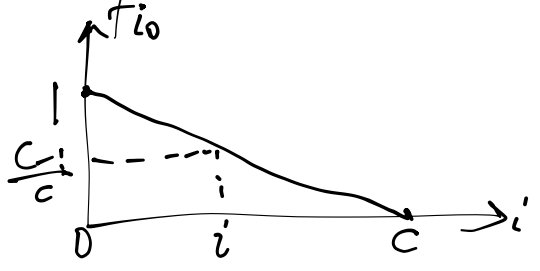
$$f_{C0} = 0$$

Special case: $p = 1/2$.

$$f_{i0} = \begin{cases} \frac{1}{2} + \frac{1}{2} f_{20} & (i=1) \\ \frac{1}{2} f_{i-1} + \frac{1}{2} f_{i+1} & (i \geq 2) \end{cases}$$

Arithmetic progression.

$$f_{i0} = \frac{C-i}{C}.$$



$$E_a[X_\tau] = c \cdot \frac{a}{c} + 0 \cdot \frac{c-a}{c} = a.$$

τ = hitting time of $\{0, c\}$

$p \neq 1/2$.

$$f_{i0} = (1-p) \cdot f_{(i-1)0} + p \cdot f_{(i+1)0}$$

$$f_{(i+1)0} - f_{i0} = \frac{1}{p} f_{i0} - \frac{(1-p)}{p} f_{(i-1)0} - f_{i0}$$

$$= \frac{1-p}{p} \cdot \left(f_{i0} - f_{(i-1)0} \right)$$

$$f_{(i+1)0} - f_{i0} = \left(\frac{1-p}{p} \right)^i \left(f_{10} - f_{00} \right)$$

$$f_{a0} = \frac{\left(\frac{1-p}{p}\right)^c - \left(\frac{1-p}{p}\right)^a}{\left(\frac{1-p}{p}\right)^c - 1}$$

Communicating states & Irreducibility.

Def. State i communicates to state j "$i \to j$"

if $f_{ij} > 0$

Def. Markov chain irreducible if $i \to j$ $(i, j \in S)$.

Def. "$i \leftrightarrow j$" means $i \to j$ and $j \to i$

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eg. Simple random walk (1-D)

Frog walk

Ehrenfest Urn

Multi-dim random walk (select random coordinate, and move)

Gambler's ruin — not irreducible.

Fact: $i \leftrightarrow k$, then i recurrent $\iff$ k recurrent.

Corollary: For irreducible MC, one of two happens.

- all states are recurrent
- all states are transient.