STA447 Lecture 4

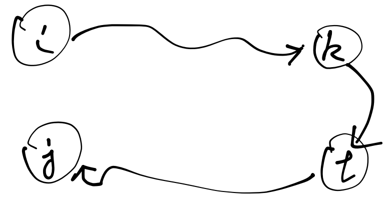
Fact. If $i \longleftrightarrow k$. then i recurrent $\Longleftrightarrow$ k recurrent.

"Case thm". two cases for irreducible chains

Proof of the fact.

Sum lemma. If $i \to k$, $l \to j$ then

$$\sum_{n=1}^{+\infty} p_{kl}^{(n)} = +\infty \Longrightarrow \sum_{n=1}^{+\infty} p_{ij}^{(n)} = +\infty.$$



$\exists\, m$ s.t. $p_{ik}^{(m)} > 0$

$\exists\, r$ s.t. $p_{lj}^{(r)} > 0$

$n > m+r$ $\quad p_{ij}^{(n)} \geq p_{ik}^{(m)} \cdot p_{kl}^{(n-m-r)} \, p_{lj}^{(r)}$

$$\sum_{n=1}^{+\infty} p_{ij}^{(n)} \geq \sum_{n=m+r+1}^{+\infty} p_{ij}^{(n)} \geq \underbrace{p_{ik}^{(m)} \cdot p_{lj}^{(r)}}_{>0} \cdot \sum_{n=m+r+1}^{+\infty} p_{kl}^{(n-m-r)}$$

Proof of "fact": Let $j=i$, $l=k$

Thm (Finite space thm).

If $|S| < +\infty$, irreducible, then all states are recurrent.

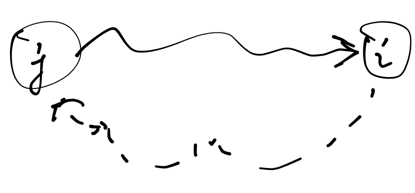
$\forall i,j \qquad \sum_{n=1}^{+\infty} P_{ij}^{(n)} \overset{?}{=} +\infty.$

Fix i, sum over j

$$\sum_{j\in S}\left(\sum_{n=1}^{+\infty} P_{ij}^{(n)}\right) = \sum_{n=1}^{+\infty}\left(\sum_{j\in S} P_{ij}^{(n)}\right) = \sum_{n=1}^{+\infty} 1 = +\infty$$

$\exists j^*$, s.t. $\sum_{n=1}^{+\infty} P_{ij^*}^{(n)} = +\infty.$

"f-Lemma": $j \to i$ and $f_{jj} = 1$ then $f_{ij} = 1$.

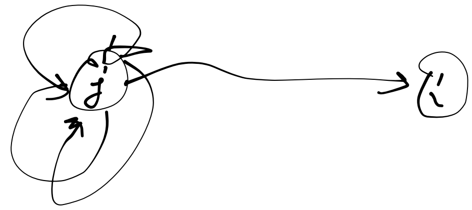


"Hit · Lemma"

$H_{ij} := \{$ we hit state i before returning to j $\}$

If $j \to i$ then $\mathbb{P}_j(H_{ij}) > 0$.

Proof of the lemma:

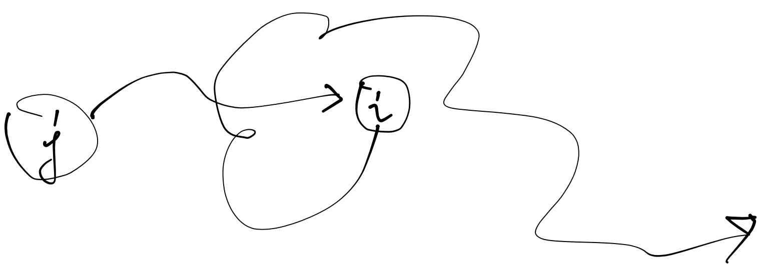


$$\begin{array}{ccccc} X_0 & X_1 & X_2 \cdots & X_\ell & X_m \\ \| & & & \| & \| \\ j & & & j & i. \end{array}$$

$$P_{x_\ell x_{\ell+1}} P_{x_{\ell+1} x_{\ell+2}} \cdots P_{x_{m-1} x_m} > 0.$$

Proof of f-lemma: $\mathbb{P}_j(H_{ij}) > 0$.

$$\mathbb{P}_j(H_{ij}) \cdot \mathbb{P}_i(\text{never visit } j) \le \underbrace{\mathbb{P}_j(\text{never return to } j)}_{=0}.$$



$$1 - f_{ij} = \mathbb{P}_i(\text{never visit } j) = 0.$$

"Infinite returns lemma":

For irreducible MC,

- Recurrent, then $\forall i, j \in S$, $\mathbb{P}_i(N_j = +\infty) = 1$
- Transient, then $\forall i, j \in S$, $\mathbb{P}_i(N_j = +\infty) = 0$.

Proof: $f_{jj}=1 \quad (j\in S) \xrightarrow{f\text{-lemma}} f_{ij}=1 \quad (\forall i,j \in S)$.

$\forall k \quad \mathbb{P}_i(N(j)\geq k) = \mathbb{P}_i(N(j)\geq k-1)\cdot \mathbb{P}_j(\text{visit back to } j)$

$$= f_{ij}\cdot (f_{jj})^{k-1} = 1.$$

So $\mathbb{P}_i(N(j)=+\infty)=1$

If chain is transient. $f_{jj}<1 \quad (\forall j\in S)$

$\mathbb{P}_i(N(j)\geq k) = f_{ij}(f_{jj})^{k-1} \qquad \mathbb{P}(N(j)<+\infty)=1.$

Recurrence Equivalence Thm.

If a MC is irreducible, the following are equi

(1) $\exists\, k,l\in S, \quad \sum_{n\geq 1}^{+\infty} p_{kl}^{(n)} = +\infty$

(2) $\forall\, k,l\in S, \quad \sum_{n\geq 1}^{+\infty} p_{kl}^{(n)} = +\infty$

(3) $\exists\, k, \quad f_{kk}=1$

(4) $\forall\, i\in S \quad f_{ii}=1$

(5) $\forall\, i,j, \quad f_{ij}=1$

(1)(2)

Recurrence State thm

(3 4 5)

Infinite returns Lemma

(6)

(6) $\exists\, k, l \in S$ $\quad \mathbb{P}_k(N(l) = +\infty) = 1$

(7). $\forall\, i, j \in S$ $\quad \mathbb{P}_i(N(j) = +\infty) = 1$

Proof: (1) $\Rightarrow$ (2) : sum Lemma

(2) $\Rightarrow$ (4) ; Recurrence state thm.

(4) $\Rightarrow$ (5) : f - lemma

(5) $\Rightarrow$ (3) : obvious.

(3) $\Rightarrow$ (1) : Recurrence state thm

(4) $\Rightarrow$ (7) : Infinite returns Lemma

(7) $\Rightarrow$ (6) : obvious

(6) $\Rightarrow$ (4) : Infinite returns lemma.

"Transience Equiv thm"

(1) $\forall\, i, j$, $\quad \sum_{n=1}^{+\infty} P_{ij}^{(n)} < +\infty$

(2). $\exists\, i, j$, $\quad \sum_{n=1}^{+\infty} P_{ij}^{(n)} < +\infty$

(3) $\forall\, k$, $\quad f_{kk} < 1$

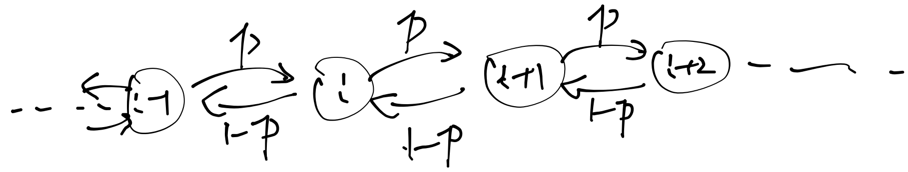
(4) $\exists\, j$ $\quad f_{jj} < 1$

(5) $\forall\, i, j$ $\quad f_{ij} < 1$

(6) $\forall\, i, j$. $\quad \mathbb{P}_i(N(j) = +\infty) = 0$

(7) $\exists\, i, j$, $\quad \mathbb{P}_i(N(j) = +\infty) = 0$.

Prop. $\exists$ irreducible MC. s.t. transient but $f_{kl}=1$ for some $k,l\in S$.



$S=\mathbb{Z}$, $\quad p>1/2$.

$$P_{00}^{(n)} = \begin{cases} 0 & (n \text{ odd}) \\ \binom{n}{n/2}\cdot p^{n/2}(1-p)^{n/2} \approx (4p(1-p))^{n/2}\cdot\sqrt{\dfrac{2}{\pi n}} & \end{cases}$$

$\left(X\sim \text{Binomial}(n,p),\ \mathbb{P}(X=n/2)\right)$.

$$\sum_{n\geq 1}^{+\infty} P_{00}^{(n)} < +\infty.$$

$\forall i<j \qquad f_{ij}=1.$

$(X_n)_{n=1}^{+\infty}$ be biased RW $\quad Z_n = X_n - X_{n-1}$ iid $\begin{cases} p & +1 \\ 1-p & -1. \end{cases}$

By LLN, $\quad \frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{a.s.} \mathbb{E}[Z]=2p-1>0.$

$$\mathbb{P}\left(\lim_{n\to\infty}\sum_{i=1}^{n} Z_i = +\infty\right)=1. \Rightarrow \mathbb{P}_i\left(\lim_{n\to+\infty} X_n=+\infty\right)=1.$$

Non-irreducible MC's

Finite state space.

$$P = \left[\begin{array}{c|c} \begin{matrix} P_1 & 0 & 0 & 0 \\ 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 \cdots & P_r \end{matrix} & 0 \\ \hline S & Q \end{array}\right]$$