STA 447/2006 Lecture 5.

Reducible Markov chains

Thm. (for finite state space)

$$\text{Transition matrix} \quad \begin{array}{c|c|l} \hline P & 0 & \text{Recurrent} \\ \hline S & Q & \text{Transient} \\ \hline \end{array} \qquad P = \begin{bmatrix} P_1 & 0 & 0 & \cdots 0 \\ 0 & P_2 & & 0 \\ 0 & 0 & \ddots & 0 \\ & 0 & & P_r \end{bmatrix}$$

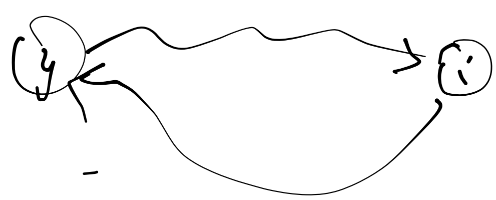
$$\left[\begin{array}{c|c} P^n & 0 \\ \hline S_n & Q^n \end{array}\right]$$

for finite state space

$Q^n \to 0.$

Fact: i transient j recurrent $j \not\to i$

Proof: Suppose $j \to i$.



$$0 = \mathbb{P}_j(\text{never return to } j)$$

$$\geq \underbrace{\mathbb{P}_j(\text{visit } i)}_{>0.} \cdot \underbrace{\mathbb{P}_i(\text{not return to } j)}_{=0.}$$

$i \leftrightarrow j$ $\quad j$ recurrent $\Rightarrow i$ recurrent.

Convergence / stationary.

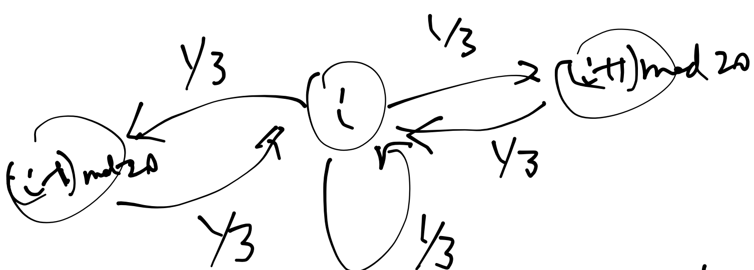
$$\lim_{n\to+\infty} \mathbb{P}(X_n = j).$$

suppose $\mu_j^{(n)} := \mathbb{P}(X_n = j)$ $\quad \mu_j^{(n)} \to q_j \ (\forall j)$

Then $\left.\begin{array}{l} \mu_j^{(n+1)} \to q_j. \\ \mu^{(n+1)} = \mu^{(n)} \cdot P \end{array}\right\} \to q = qP.$

Def. Say π is stationary distribution when $\pi = \pi P$.

eg. Frog walk. $\pi = \left[\frac{1}{20}, \frac{1}{20}, \cdots, \frac{1}{20}\right]$.



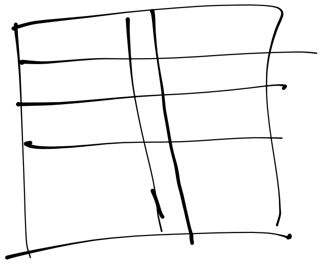
$$\frac{1}{20} = \pi_i = \sum_j \pi_j P_{ji} = \frac{1}{20} \cdot \frac{1}{3} \cdot 3 = \frac{1}{20}$$

π_1 on S_1

eg. $P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{array}{l} \}\, S_1 \\ \}\, S_2 \end{array}$ $\quad$ π_2 on S_2.

eg. Doubly stochastic matrices.

$$\forall i \quad \sum_{j \in S} P_{ij} = 1.$$

$P =$ 

$$\text{If } \forall j \quad \sum_{i \in S} P_{ij} = 1$$

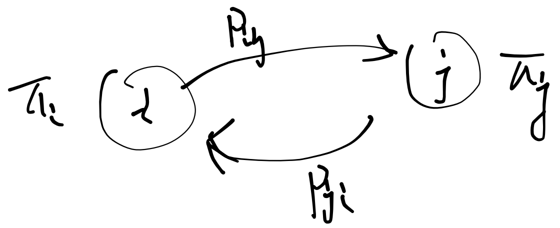
then we call it doubly stochastic.

Fact: uniform is stationary for doubly stochastic P.

$$\frac{1}{|S|} = \pi_i \overset{?}{\underset{\checkmark}{=}} \sum_{j \in S} \pi_j P_{ji} = \frac{1}{|S|} \sum_{j \in S} P_{ji} = \frac{1}{|S|}$$

Def. Reversible / detailed balance. with respect to π

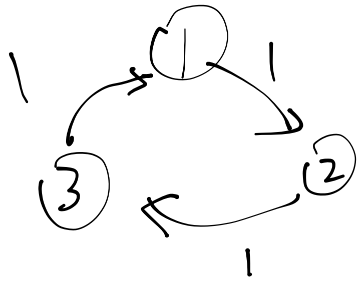
if $\quad \pi_i P_{ij} = \pi_j P_{ji} \quad (\forall i, j \in S).$



Prop. P reversible w.r.t. π then π is stationary for P.

Proof. $$\sum_{i \in S} \pi_i P_{ij} = \sum_{i \in S} \pi_j P_{ji} = \pi_j \quad (\forall j \in S).$$

Fact. $\exists$ Markov chain P w/ stationary π s.t. P is not reversible w.r.t. π.

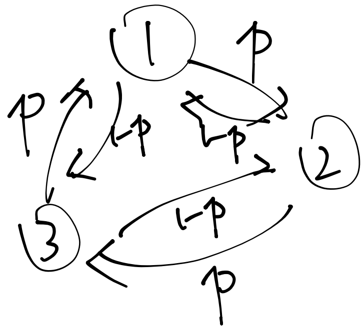


$\pi_i = 1/3 \quad \forall i \in S.$

$\pi_i = \sum_{j \in S} \pi_j P_{ji}$

$1/3 = \pi_1 P_{12} \neq \pi_2 P_{21} = 0.$

eg.



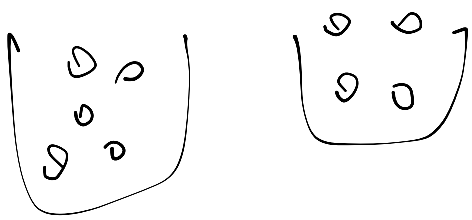
$\forall p \in [0,1].$

$\pi_i = 1/3 \quad \forall i \in S$

reversible only when $p = 1/2$.

eg. Ehrenfest's Urn

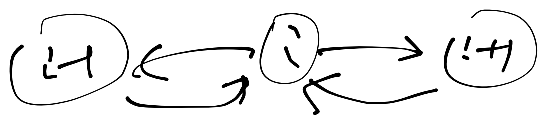
d balls in total



Idea: each ball equally likely to be in both boxes.

Conjecture: $\pi_i = \binom{d}{i} 2^{-d}$ i.e. $\text{Binom}(d, 1/2)$.

Verify detailed balance



$$\pi_i P_{i(i+1)} = 2^{-d} \cdot \binom{d}{i} \cdot \frac{d-i}{d} = 2^{-d} \cdot \frac{d!}{(d-i)!\, i!} \cdot \frac{d-i}{d}$$
$$= 2^{-d} \cdot \frac{(d-1)!}{i!\,(d-i-1)!}.$$

$$\pi_{i+1} P_{(i+1)i} = 2^{-d} \cdot \frac{d!}{(i+1)!(d-i-1)!} \cdot \frac{i+1}{d} = 2^{-d} \frac{(d-1)!}{i!(d-i-1)!}$$