STA447/2006 Lecture 6.

$\pi = \pi P$ Reversible: $\pi_i p_{ij} = \pi_j p_{ji}$

SRW? Stationary does not exist.

Vanishing probabilities proposition.

If $\lim_{n\to+\infty} p_{ij}^{(n)} = 0$ for all $i,j \in S$

then stationary distribution does not exist

Proof: (by contradiction) Suppose that π is stationary

$\forall j \in S$ $\pi_j = \sum_{i\in S} \pi_i p_{ij} = \sum_{i\in S} \pi_i p_{ij}^{(n)}$ $(\forall n \in \mathbb{N}_+)$

$(\pi = \pi P = \pi P^2 \cdots = \pi P^n)$

$$\pi_j = \lim_{n\to+\infty} \sum_{i\in S} \pi_i p_{ij}^{(n)} \overset{?}{=} \sum_{i\in S} \underbrace{\lim_{n\to+\infty} \pi_i p_{ij}^{(n)}}_{0} = 0$$

(Detour) M-test

$\{x_{nk}\}_{n,k\in\mathbb{N}}$, suppose $\lim_{n\to+\infty} x_{nk}$ exists for each $k \in \mathbb{N}$, and

$$\sum_{k=1}^{+\infty} \sup_{n\geq 1} |x_{nk}| < +\infty \quad (*)$$

Then $\lim_{n\to+\infty} \sum_{k=1}^{+\infty} x_{nk} = \sum_{k=1}^{+\infty} \lim_{n\to+\infty} x_{nk}$

Check: $\sum_{i\in S} \sup_{n\geq 1} |\pi_i P_{ij}^{(n)}| \leq \sum_{i\in S} \pi_i = 1$

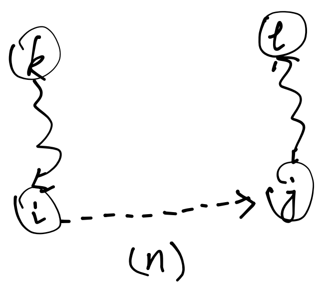
$\pi_j = 0 \quad \forall j \in S$ contradiction.

"Vanishing Lemma": If $\lim_{n\to+\infty} P_{kl}^{(n)} = 0$ for some $k, l \in S$

and for $i, j \in S$, $k \to i$, $j \to l$

then $\lim_{n\to+\infty} P_{ij}^{(n)} = 0$

Proof:



$\exists\, r, s \geq 0 \qquad P_{ki}^{(r)} > 0, \ P_{jl}^{(s)} > 0$

$$\underbrace{P_{kl}^{(r+n+s)}}_{\substack{\downarrow (n\to+\infty) \\ 0}} \geq P_{ki}^{(r)} \cdot \underbrace{P_{ij}^{(n)}}_{\substack{\downarrow \\ 0}} \cdot P_{jl}^{(s)} \geq 0 \quad \text{as } (n\to+\infty).$$

Corollary: For an irreducible MC, if $\exists\, i, j \in S$ $\lim_{n\to+\infty} P_{ij}^{(n)} = 0$ then stationary does not exist.

Irreducible MC $\begin{cases} \lim_{n\to+\infty} P_{ij}^{(n)} = 0 \ (\forall i, j) & \text{stationary does not exist} \\ P_{ij}^{(n)} \not\to 0 \ (\forall i, j \in S). \end{cases}$

e.g. 1-D SRW, $p_{00}^{(n)} \sim \frac{C}{\sqrt{n}} \to 0$

e.g. Irreducible & Transient MC

$$\sum_{n\geq 1} p_{ij}^{(n)} < +\infty \qquad (\forall i,j \in S)$$

$$\lim_{n\to\infty} p_{ij}^{(n)} = 0$$

Remark: $\sum_{j\in S} p_{ij}^{(n)} = 1 \qquad (\forall n,\ \forall i \in S)$

$\lim_{n\to+\infty} p_{ij}^{(n)} = 0 \qquad (\forall i,j)$

not a contradiction

$$\sum_{j} \sup_{n\geq 1} p_{ij}^{(n)} = +\infty$$

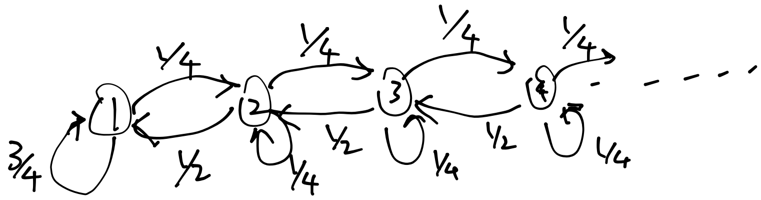
e.g. For finite space MC $\quad v = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$Pv = v \qquad$ 1 eigen value, v right eigen-vector

$\exists$ left eigen vector.

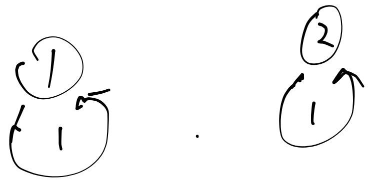
For infinite space MC, stationary can exist.

e.g. $\quad S = \{1, 2, 3, \cdots\}$



$\pi_i = 2^{-i} \qquad 2^{-(i+2)} = \pi_i P_{i(i+1)} = \pi_{i+1} P_{(i+1)i} = 2^{-(i+2)}$

"Non-convergence".

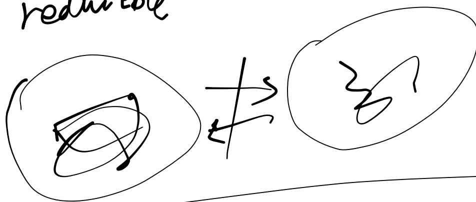


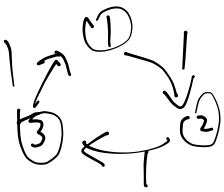
$\pi_1 = \pi_2 = 1/2$

$\pi_1 = 1,\ \pi_2 = 0$

(π_1, π_2) $\left(\begin{array}{l} \pi_1 + \pi_2 = 1 \\ \pi_1, \pi_2 \geq 0 \end{array}\right)$ stationary.

reducible





$$\mathbb{P}_1(X_n = 1) = \begin{cases} 1 & n \equiv 0 \pmod 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\pi_1 = \pi_2 = \pi_3 = 1/3.$$

$$p_{ij}^{(n)} \not\to \pi_j$$

Def. Period of state $i \in S$ is the greatest common divisor (gcd) of the set $\{n \geq 1 : p_{ii}^{(n)} > 0\}$

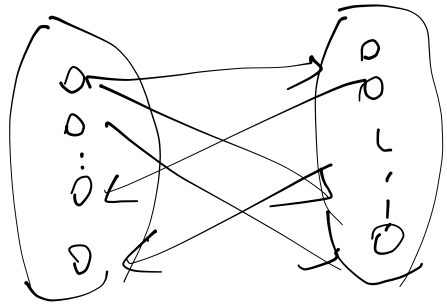
If period of state i is 1, we call it aperiodic

(e.g. $\gcd(6, 15, 21) = 3$)

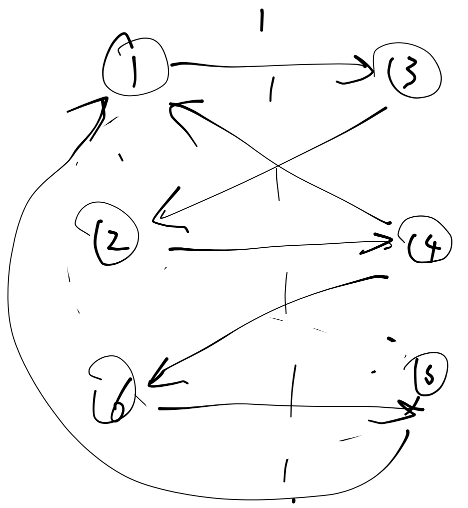
e.g. $\{n \geq 1 : p_{11}^{(n)} > 0\} = \{3, 6, 9, \dots\}$

Periods of 1, 2, 3 are 3

e.g.



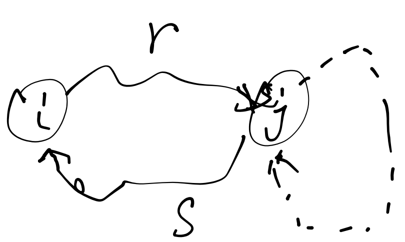
period of any state is at least 2.



e.g. $p_{ii} > 0$ then i is aperiodic

e.g. $p_{ii}^{(n)} > 0$, $p_{ii}^{(n+1)} > 0$, then i . . . . . .

Lemma. (Equal period). $i \leftrightarrow j$, then they have equal period.



$p_{ij}^{(r)} > 0$

$p_{ji}^{(s)} > 0$

t_i period of i

t_j period of j

$$p_{ii}^{(r+s)} \geq p_{ij}^{(r)} \cdot p_{ji}^{(s)} > 0 \qquad t_i \mid (r+s)$$

Suppose $p_{jj}^{(n)} > 0$ for some n

$p_{ii}^{(r+n+s)} \geq p_{ij}^{(r)} \cdot p_{jj}^{(n)} \cdot p_{ji}^{(s)} > 0$ $\quad t_i \mid (r+n+s)$

$t_i \mid n$ $\quad (\forall n$ s.t. $p_{jj}^{(n)} > 0)$

$t_i \mid t_j$ also $t_j \mid t_i$ so $t_i = t_j$.

Corollary: For irreducible MC, all states have the same period.

Thm (MC convergence)

If a MC is irreducible, aperiodic, and has a stationary distribution π, then

$$\forall i, j \in S, \quad \lim_{n \to +\infty} p_{ij}^{(n)} = \pi_j.$$

Furthermore, starting from any ν,

$$\lim_{n \to +\infty} \mathbb{P}(X_n = j) = \pi_j$$

Basic properties $\begin{cases} \exists \text{ stationary} + \text{irreducibility} \Rightarrow \text{recurrence} \\ \text{Aperiodicity} \end{cases}$

Prop. If i is aperiodic and $f_{ii} > 0$

then $\exists\, n_0(i) \in \mathbb{N}$, s.t. $p_{ii}^{(n)} > 0$ $(\forall n \geq n_0)$

$A = \{ n \geq 1 : P_{ii}^{(n)} > 0 \}$ $\quad \gcd(A) = 1$

"Additivity : $m \in A,\ n \in A \implies m+n \in A$"

$$P_{ii}^{(m)} > 0,\ P_{ii}^{(n)} > 0 \implies P_{ii}^{(m+n)} \geq P_{ii}^{(m)} \cdot P_{ii}^{(n)} > 0.$$

Additivity + Bézout's identity $\implies$ Lemma

Corollary: (Irreducible) $\forall i,j \in S$, $\exists\, n_0(i,j) \in S$ s.t. $P_{ij}^{(n)} > 0$, $\forall\, n > n_0(i,j)$

$\exists\, m$, s.t. $P_{ij}^{(m)} > 0$,

$\forall n \geq n_0$ $\quad P_{ii}^{(n)} > 0$

$$P_{ij}^{(n+m)} \geq P_{ii}^{(n)} \cdot P_{ij}^{(m)} > 0.$$

Key Lemma (Markov forgetting)

Under same conditions. $\forall i,j,k \in S$

$$\lim_{n \to +\infty} \left| P_{ik}^{(n)} - P_{jk}^{(n)} \right| = 0.$$