STA 447/2006 Lecture 7.

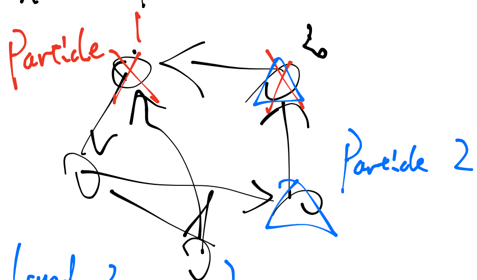
Thm If a Markov chain is irreducible, aperiodic, and has a stationary

then $\lim_{n\to+\infty} p_{ij}^{(n)} = \pi_j$ $(\forall i,j \in S)$.

Markov forgetting lemma. under same assumptions,

$\forall i,j,k \in S$ $\quad \lim_{n\to+\infty} \left| p_{ik}^{(n)} - p_{jk}^{(n)} \right| = 0.$

Proof: "coupling"



$\tau = \inf\{ t>0 : \text{Particles 1 and 2 meets} \}$

A new MC with state space $\bar{S} = S \times S$ $\quad \{(X_n, Y_n)\}_{n=1}^{+\infty}$

$$\bar{P}_{(i,j),(k,l)} = \mathbb{P}(X_1 = j, Y_1 = l \mid X_0 = i, Y_0 = k)$$

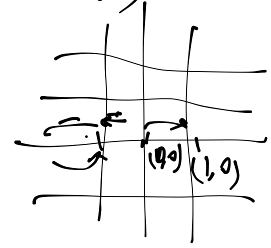
$$= \mathbb{P}(X_1 = j \mid X_0 = i) \cdot \mathbb{P}(Y_1 = l \mid Y_0 = k)$$

$$= p_{ik} \cdot p_{jl}.$$

— has a stationary distribution

$\bar{\pi}_{(i,j)} = \pi_i \cdot \pi_j$ (easy to verify)

— Irreducibility from aperiodicity



From last lecture, $\forall i,j \in S$, $\exists n_0(i,j) \in \mathbb{N}$

s.t. for $n \geq n_0(i,j)$ $p_{ij}^{(n)} > 0$.

$n \geq \max\{ n_0(i,k), n_0(j,l) \}$

$$P_{(ij),(kl)}^{(n)} = p_{ik}^{(n)} \, p_{jl}^{(n)} > 0.$$

New chain is irreducible and aperiodic

$\longrightarrow$ "Stationary recurrence theorem" $\Longrightarrow$ recurrent.

Consider $\forall i_0$, $\tau := \inf\{ n \geq 0 : X_n = Y_n = i_0 \}$

"hitting time for (i_0, i_0) in the new MC"

recurrence $\Rightarrow \mathbb{P}_{ij}(\tau < +\infty) = 1$.

$$p_{ik}^{(n)} = \mathbb{P}_i(X_n = k) = \mathbb{P}_{ij}(X_n = k)$$

$$= \sum_{m=1}^{+\infty} \mathbb{P}_{ij}(X_n = k, \tau = m).$$

$$= \underbrace{\sum_{m=1}^{n} \mathbb{P}_{ij}(X_n = k, \tau = m)}_{\text{Hit } (i_0,i_0) \text{ before } n} + \underbrace{\mathbb{P}_{ij}(X_n = k, \tau > n)}_{\text{Hit } (i_0,i_0) \text{ after } n}$$

$(m \leq n)$

$$\mathbb{P}_{ij}(X_n = k, \tau = m) = \mathbb{P}_{ij}(\tau = m) \cdot \mathbb{P}_{ij}(X_n = k \mid \tau = m)$$

$$= \mathbb{P}_{ij}(\tau = m) \cdot \mathbb{P}(X_n = k \mid X_m = i_0)$$

$\mathbb{P}(Y_n = k \mid \tau = m) = \mathbb{P}(Y_n = k \mid Y_m = i_0)$

$$p_{ik}^{(n)} = \boxed{\sum_{m=1}^{n} \mathbb{P}_{ij}(\tau=m)\cdot p_{i_0k}^{(n-m)}} + \mathbb{P}_{ij}(X_n=k,\ \tau>n).$$

$$p_{jk}^{(n)} = \mathbb{P}_j(Y_n=k) = \mathbb{P}_{ij}(Y_n=k)$$

$$= \boxed{\sum_{m=1}^{n} \mathbb{P}_{ij}(\tau=m)\cdot p_{i_0k}^{(n-m)}} + \mathbb{P}_{ij}(Y_n=k,\ \tau>n)$$

The same.

$$\left|p_{ik}^{(n)} - p_{jk}^{(n)}\right| = \left|\mathbb{P}_{ij}(X_n=k,\tau>n) - \mathbb{P}_{ij}(Y_n=k,\ \tau>n)\right|$$

$$\left(\le \mathbb{P}_{ij}(X_n=k,\tau>n) + \mathbb{P}_{ij}(Y_n=k,\ \tau>n)\right)$$

$$\le 2\cdot \mathbb{P}_{ij}(\tau>n)$$

$$\mathbb{P}_{ij}(\tau<+\infty)=1 \Rightarrow \lim_{n\to+\infty} \mathbb{P}_{ij}(\tau>n)=0.$$

$$\left|p_{ij}^{(n)} - \pi_j\right| = \left|p_{ij}^{(n)} - \sum_{k\in S}\pi_k p_{kj}^{(n)}\right|$$

$$= \left|\sum_{k\in S}\pi_k\left(p_{ij}^{(n)} - p_{kj}^{(n)}\right)\right|$$

$$\le \sum_{k\in S}\pi_k \underbrace{\left|p_{ij}^{(n)} - p_{kj}^{(n)}\right|}_{\to 0} \quad \text{for each } k$$

M-test

$$\sum_{k\in S}\left(\sup_{n\ge 0}\pi_k \underbrace{\left|p_{ij}^{(n)} - p_{kj}^{(n)}\right|}_{\le 1}\right) \le \sum_{k\in S}\pi_k = 1.$$

So we have $\lim_{n\to+\infty}\left|p_{ij}^{(n)} - \pi_j\right| = 0.$

Starting from initial distribution $\{\nu_i\}_{i\in S}$

$$\lim_{n\to+\infty} \mathbb{P}(X_n = j) = \lim_{n\to+\infty} \sum_{i\in S} \nu_i \cdot P_{ij}^{(n)} \overset{?}{=} \sum_{i\in S} \nu_i \cdot \lim_{n\to+\infty} P_{ij}^{(n)}$$

$$= \sum_{i\in S} \nu_i \cdot \pi_j = \pi_j$$

Justify "$\overset{?}{=}$": M-test

$$\sum_{i\in S} \sup_{n\geq 0} \left| \nu_i \, P_{ij}^{(n)} \right| \leq \sum_{i\in S} \nu_i = 1 < +\infty.$$

Corollary: If MC is irreducible and aperiodic, it has at most one stationary distribution