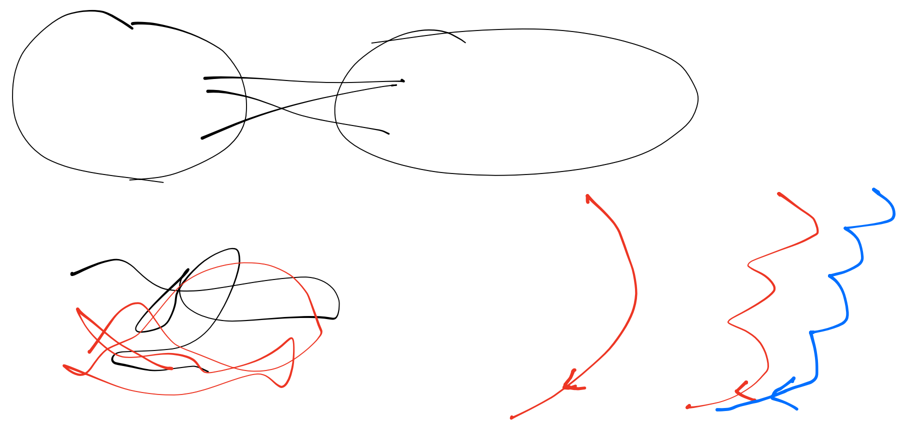
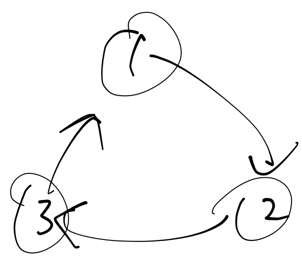
STA 447/2006 Lecture 8.



Periodic convergence.

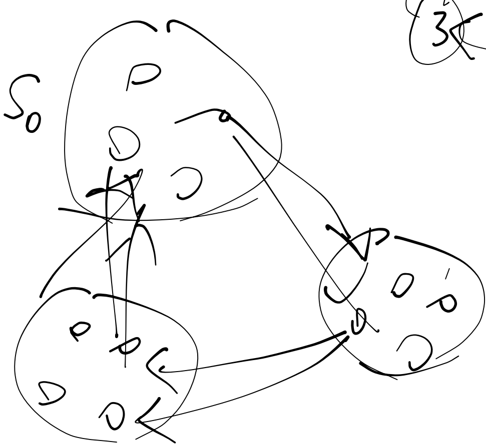


e.g.

$$P_{11}^{(n)} = \begin{cases} 1 & \frac{n}{3} \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

average 3 steps $= 1/3$

S_0



Thm. MC irreducible, with a period $b \geq 2$.

has a stationary distribution π

Then, $\forall i, j \in S$

$$\lim_{n \to +\infty} \frac{1}{b}\left[P_{ij}^{(n)} + P_{ij}^{(n+1)} + \cdots + P_{ij}^{(n+b-1)} \right] = \pi_j$$

For initial distribution ν

$$\lim_{n\to+\infty} \frac{1}{b}\Big(\mathbb{P}(X_n=j)+\mathbb{P}(X_{n+1}=j)+\cdots+\mathbb{P}(X_{n+b-1}=j)\Big)=\pi_j$$

Corollary. (Cesàro sum)

For any MC irreducible, $\exists$ stationary distr π

$\forall i,j\in S$, $\displaystyle\lim_{n\to+\infty}\frac{1}{n}\sum_{t=1}^{n} p_{ij}^{(t)} = \pi_j$

Corollary. An irreducible MC, has at most one stationary distribution.

Proof of periodic convergence.

"Cyclic decomp Lemma": period $b\geq 2$

Then $S = S_0 \cup S_1 \cup \cdots \cup S_{b-1}$ $\quad (S_i\cap S_j=\emptyset \ \forall i,j)$

s.t. If $i\in S_r$, $\{j\in S: p_{ij}>0\}\subseteq S_{(r+1)\bmod b}$

Furthermore, $P^{(b)}$ (restricted to S_0)

forms an irreducible & aperiodic transition matrix.

Proof idea:
Fix any $i_0 \in S$ $\quad S_r = \{ j \in S : P_{i_0 j}^{(bm+r)} > 0 \text{ for some } m \}$

— Irreducible $\Rightarrow$ Cover S

— Periodicity $\Rightarrow$ disjoint.

$P^{(b)}$ is well defined on S_0.

— Irreducibility: states in S_0 reachable only in $n = mb$ steps for some m

— Aperiodicity: If $P^{(b)}$ has period $m \geq 2$, then mb is a period of P.

Proof overview of periodic convergence

— $\pi(S_0) = \pi(S_1) = \cdots = \pi(S_{b-1}) = \frac{1}{b}$

$\left(\pi(S_i) := \sum_{j \in S_i} \pi_j \right)$

Take $\hat{\pi}_i := b \cdot \pi_i$ for $i \in S_0$. $\hat{\pi}$ is stationary for $P^{(b)}$

$P^{(b)}$ is irreducible & aperiodic

$$\lim_{m\to+\infty} P_{ij}^{(bm)} = \hat{\pi}_j = b\cdot\pi_j \qquad (\forall\, i,j\in S_0)$$

similar for $S_1, S_2, \ldots, S_{b-1}$

Averaging yields desired results

$$\hat{\pi} \overset{?}{=} \hat{\pi}\hat{P}^{(b)}$$

$$P^{(b)} = \begin{array}{c|ccc} & S_0 & S_1 & S_2 \\ \hline S_0 & \hat{P}^{(b,S_0)} & 0 & 0 \\ S_1 & 0 & \hat{P}^{(b,S_1)} & 0 \\ S_2 & 0 & 0 & \hat{P}^{(b,S_{b-1})} \end{array}$$

$$\pi = \pi\cdot P^{(b)}.$$

$$\pi = [\underbrace{\hat{\pi}^{(S_0)}}_{S_0} \quad \underbrace{\hat{\pi}^{(S_1)}\ \ldots}_{S_1} \quad \ldots \quad \underbrace{\hat{\pi}^{(S_{b-1})}\ \ldots}_{S_{b-1}}]$$

$$\|$$

$$\pi\cdot P^{(b)} = [\hat{\pi}^{(S_0)}\cdot\hat{P}^{(b,S_0)}, \quad \ldots\ldots \quad]$$

$\pi(S_0) = 1/b$ $\qquad$ $b\cdot\pi|_{S_0}$ is a prob distr on S_0

Know: $\displaystyle\lim_{m\to+\infty} \frac{1}{mb}\cdot\sum_{t=1}^{mb} P_{ij}^{(t)} \to \pi_j$ $\qquad$ $\displaystyle\frac{1}{m}\sum_{k=0}^{m} \underbrace{\frac{1}{b}\sum_{r=0}^{b-1} P_{ij}^{(kb+r+1)}}_{\to\ \pi_j}$

$n = mb + r \qquad 0\le r\le b-1$

$$\frac{1}{n}\sum_{t=1}^{n} P_{ij}^{(t)} = \frac{1}{n}\sum_{t=1}^{mb} P_{ij}^{(t)} + \frac{1}{n}\sum_{t=mb+1}^{n} P_{ij}^{(t)}$$

Mean recurrence time.

$$T_i = \inf\{n \geq 1 : X_n = i\}$$

Question: $\mathbb{E}_i[T_i] \overset{?}{<} +\infty$

Def. A state is $\begin{cases} \text{positive recurrent if } \mathbb{E}_i[T_i] < +\infty \\ \text{null recurrent if } \mathbb{E}_i[T_i] = +\infty \text{ but } T_i < +\infty \text{ a.s.} \end{cases}$

Roughly speaking, $\mathbb{E}_i[T_i] = \frac{1}{\pi_i}$

Consider MC starting from i.

Time $0, 1, 2, \dots, T_i, \dots, T_i^{(2)} \dots T_i^{(3)} \dots$

state $i, j, k, \dots\dots i \quad i \quad i \dots$

$T_i,\ T_i^{(2)} - T_i,\ T_i^{(3)} - T_i^{(2)}, \dots \sim$ iid.

By average convergence, $\pi_i = \lim_{n\to\infty} \boxed{\frac{1}{n}\sum_{1}^{n} P_{ii}^{(t)}}$

$= \mathbb{E}[\text{Frequency of } i]$.

By LLN, $\frac{T_i^{(k)}}{k} \to \mathbb{E}_i[T_i]$

Want to show.

$$\pi_i = \frac{1}{\mathbb{E}_i[T_i]} = \text{Frequency of } i \text{ appearing in trajectory.}$$

Detour: stationary measure.

$$\mu = \mu P \qquad \mu \geq 0$$

Thm: For any irreducible and recurrent MC,

$\forall i_0 \in S$,

$$\mu_{i_0}(y) = \sum_{n=0}^{+\infty} \mathbb{P}_{i_0}(X_n = y, T_{i_0} > n)$$

μ_{i_0} is a stationary measure.

Intuition: $\mu_{i_0}(j) = \mathbb{E}_{i_0}[\#\text{ visits to } j \text{ in time } \{0, 1, \cdots, T_{i_0} - 1\}]$.

$$\mu_{i_0} P(y) \overset{?}{=} \mathbb{E}_{i_0}[\text{---------} \{1, 2, \cdots, T_{i_0}\}]$$

"cycle trick".

Proof: $\mu_{i_0}(i_0) = 1$.

$$\sum_{j \in S} \mu_{i_0}(j) P_{jk} = \sum_{n=0}^{+\infty} \sum_{j \in S} \mathbb{P}_{i_0}(X_n = j, T_{i_0} > n) \cdot P_{jk}$$

$$\text{each term} = \begin{cases} \mathbb{P}_{i_0}(X_n = j,\ X_{n+1} = k,\ T_{i_0} > n+1) & (k \neq i_0) \\ \mathbb{P}_{i_0}(X_n = j,\ T_{i_0} = n+1) & (k = i_0) \end{cases}$$

Sum over j
$$\begin{cases} \mathbb{P}_{i_0}(X_{n+1} = k,\ T_{i_0} > n+1) & (k \neq i_0) \\ \mathbb{P}_{i_0}(T_{i_0} = n+1) & (k = i_0). \end{cases}$$

Sum over n
$$\begin{cases} \sum_{n=0}^{+\infty} \mathbb{P}_{i_0}(X_{n+1} = k,\ T_{i_0} > n+1) & (k \neq i_0) \\ 1 & (k = i_0). \end{cases}$$

$$= \mu_{i_0}(k).$$

Check $\mu_{i_0}(j) < +\infty$ $(\forall n > 0)$.

$$\mu_{i_0}(i_0) = 1 = \sum_{j \in S} \mu_{i_0}(j) \cdot p^{(n)}(j, i_0) \geq \mu_{i_0}(j) \cdot \underbrace{p^{(n)}(j, i_0)}_{>0}$$

$$\mu_{i_0}(j) \leq \frac{1}{p^{(n)}(j, i_0)} < +\infty.$$

Check $\mu_{i_0}(j) > 0$ $(\forall j \in S)$. (Hit Lemma).