STA 447/2006 Lecture 9.

Thm P irreducible, recurrent

$$\mu_x(y) := \sum_{n=0}^{+\infty} \mathbb{P}_x(X_n = y,\ T_x > n)$$

- $0 < \mu_x(y) < +\infty \quad \forall x, y \in S$
- $\mu = \mu P$.

When does it become a stationary distribution?

$$\sum_{j \in S} \mu_i(j) = \sum_{j \in S} \sum_{n=0}^{+\infty} \mathbb{P}_i(X_n = j,\ T_i > n)$$

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$$= \sum_{n=0}^{+\infty} \mathbb{P}_i(T_i > n)$$

$$= \mathbb{E}_i[T_i]$$

$\mu_i(i) = 1$ if $\mathbb{E}_i[T_i] < +\infty$, then we can normalize

$\pi = \dfrac{\mu_i}{\mathbb{E}_i[T_i]}$ is a stationary distribution.

Corollary: irreducible If $\exists$ state i positive recurrent, then stationary distribution exists.

The other way?

Thm. P irreducible and recurrent $N_n(i) := \sum_{t=1}^{n} \mathbb{1}\{X_t = i\}$

the $\frac{N_n(i)}{n} \longrightarrow \frac{1}{\mathbb{E}_i[T_i]}$ a.s.

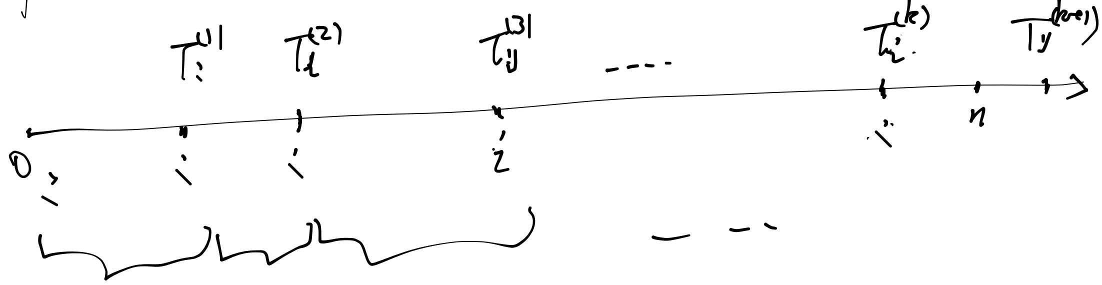
Corollary: P irreducible w/ stationary distribution π

then $\pi_i = \frac{1}{\mathbb{E}_i[T_i]}$ $(\forall i \in S)$

Proof of Corollary: $\frac{\mathbb{E}_i[N_n(i)]}{n} = \frac{1}{n} \sum_{t=1}^{n} p_{ii}^{(t)} \longrightarrow \pi_i$

$T_i^{(k)} :=$ k-th visit to i

Proof of thm.



For $k = 0, 1, \dots$ $\{X_t : T_i^{(k)} < t \leq T_i^{(k+1)}\}$ are iid.

$$\frac{T_i^{N_n(i)}}{N_n(i)} \leq \frac{n}{N_n(i)} \leq \frac{T_i^{(N_n(i)+1)}}{N_n(i)}$$

$$\frac{T_i^{(k)}}{k} = \frac{1}{k}\sum_{l=1}^{k}\left(T_i^{(l)} - T_i^{(l-1)}\right) \xrightarrow{SLLN} \mathbb{E}_i[T_i] \quad \text{(a.s.)}$$

$$\begin{cases} \text{when } \mathbb{E}_i[T_i] < +\infty \\ \text{when } \mathbb{E}_i[T_i] = +\infty \end{cases}$$

$$\frac{T_i^{(k)}}{k} \to +\infty \quad \text{(a.s.)}$$

$$\frac{T_i^{(k+1)}}{k+1} \xrightarrow{a.s.} \mathbb{E}_i[T_i]$$

$$\frac{T_i^{(k+1)}}{k} \xrightarrow{a.s.} \mathbb{E}_i[T_i]$$

$$\mathbb{P}\left(N_n(i) \to +\infty\right) = 1.$$

Thm (case theorem for positive recurrence)

For recurrent, irreducible P, only two cases can happen

(i) $\mathbb{E}_i[T_i] < +\infty$ $\forall i \in S$, $\exists$ unique stationary distribution

$$\pi_i = \frac{1}{\mathbb{E}_i[T_i]}$$

(ii) $\mathbb{E}_i[T_i] = +\infty$ $\forall i \in S$, there is no stationary distribution.

Thm. P irreducible, positive recurrent, stationary π,

For any function f, s.t. $\mathbb{E}_\pi[|f|] < +\infty$

then $\frac{1}{n}\sum_{t=1}^{n} f(X_t) \to \mathbb{E}_\pi[f]$. a.s.

Proof (based on decomp) Fix $i \in S$, define $T_k := T_i^{(k)}$

$$Y_k = \sum_{t=T_{k-1}+1}^{T_k} f(X_t)$$

Key obs: $\mathbb{E}[Y_k] = \sum_y \mu_x(y) f(y)$ — apply SLLN