STA447/2006: Midterm Exam #1

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This exam contains 10 pages.

Total marks: 100 pts

Time Allowed: 105 minutes

Question 1. [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

T (1) For three states i, j, k, if $f_{ij} > 0$ and $p_{jk} > 0$, then $f_{ik} > 0$.

F (2) If i is transient and j is recurrent, then $f_{ij} < 1$.

F (3) Let P be an irreducible and transient Markov chain. Then for any pair of states $i, j \in S$, we have $f_{ij} < 1$.

T (4) Let P be an irreducible and recurrent Markov chain. Then for any pair $i, j \in S$, with probability 1, the Markov chain starting from i will visit j infinitely often.

T (5) Let P be an irreducible Markov chain. Suppose that the stationary distribution π exists, then we have $\pi_i > 0$ for any $i \in S$.

F (6) Let P be a reducible but aperiodic Markov chain. Suppose that P has at least one stationary distribution. For any $i \in S$, there must exist a stationary distribution $\pi^{(i)}$ of P, such that $\lim_{n\to+\infty} p_{ij}^{(n)} = \pi_j^{(i)}$.

F (7) Let P be a Markov transition kernel, if P and P^2 are both irreducible, then P^3 is also irreducible.

T (8) There exists an irreducible Markov chain P with a period $b \geq 2$, but $p_{ii}^{(b)} = 0$ for any $i \in S$.

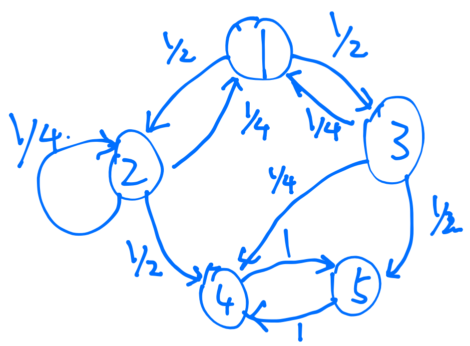
F (9) If a Markov chain P does not have a stationary distribution, then for any pair of states $i, j \in S$, we have $\mathbb{E}_i[T_j] = +\infty$.

T (10) Let P be a finite-state Markov chain on state space S. If $p_{ij}^{(k)} = 0$ for $k = 1, 2, \cdots, |S| - 1$, then $i \nrightarrow j$ (i.e., it is impossible to go from i to j).

Question 2. [22 pts] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/2 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

(1) [5 pts]. Which states are recurrent? Which states are transient? Please explain your reasoning.



Recurrent: $\{4,5\}$

Transient: $\{1,2,3\}$

Reasoning:

From 4, you must return to 4 in 2 steps; similar for 5. So {4,5} are recurrent states.

From one of the states in {1,2,3}, with positive probability, the process will hit {4,5} within 3 steps, and never return to the starting state. So {1,2,3} are transient states.

(2) [10 pts]. Compute f_{12} and f_{32}. Explain your reasoning.

By f - expansion,

$$(*)\begin{cases} f_{12} = \frac{1}{2} + \frac{1}{2} f_{32} \\ f_{32} = \frac{1}{4} f_{12} + \frac{1}{4} f_{42} + \frac{1}{2} f_{52} \end{cases}$$

Impossible to go from 4 or 5 ~~to~~ 2, so $f_{42} = f_{52} = 0$.

Solving $(*)$,
$$\begin{cases} f_{12} = \frac{4}{7} \\ f_{32} = \frac{1}{7} \end{cases}$$

(3) [7 pts]. Find a stationary distribution π of P, and show that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{t=1}^{n} p_{1j}^{(n)} = \pi_j, \quad \text{for any } j \in S.$$

$\pi = \left[0,\ 0,\ 0,\ \frac{1}{2},\ \frac{1}{2}\right]$ is stationary.

Define $\tau_j :=$ hitting time for j. $\quad \mathbb{P}_1(\tau_4 < +\infty) = \mathbb{P}_1(\tau_5 < +\infty) = 1$

$$\frac{1}{n}\sum_{t=1}^{n} P_{1j}^{(t)} = \frac{1}{n}\sum_{t=1}^{\tau_4} P_{1j}^{(t)} + \frac{1}{n}\sum_{t=\tau_4+1}^{n} P_{1j}^{(t)}$$

$$\left|\frac{1}{n}\sum_{t=1}^{\tau_4} P_{1j}^{(t)}\right| \leq \frac{\tau_4}{n} \longrightarrow 0 \quad (a.s.).$$

$$\frac{1}{n}\sum_{t=\tau_4+1}^{n} P_{1j}^{(t)} = \frac{n-\tau_4}{n} \cdot \frac{1}{n-\tau_4}\sum_{t=1}^{n-\tau_4} P_{4j}^{(t)} \longrightarrow \begin{cases} 0 & j \in \{1,2,3\} \\ \frac{1}{2} & j \in \{4,5\} \end{cases}$$

So we conclude that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{t=1}^{n} P_{1j}^{(t)} = \pi_j \quad \text{for } j \in S$$

Question 3. [30 pts, 10 pts each] For each of the following statement, *either prove it or give a counter-example.* Please provide a complete justification. Your grade will be based on the reasoning. You will receive zero points if you only give true-or-false answers without justification.

Statement (1). Let P be a Markov chain and let j be a transient state. For any other state $i \in S$, we have

$$\sum_{n\geq 0} p_{ij}^{(n)} \leq \sum_{n\geq 0} p_{jj}^{(n)},$$

here we use the convention $p_{jj}^{(0)} = 1$ and $p_{ij}^{(0)} = 0$ for $i \neq j$.

True.

For $i \neq j$ $\quad \sum_{n\geq 0} p_{ij}^{(n)} = \sum_{n\geq 0} \mathbb{P}_i(N(j) \geq n) = \frac{f_{ij}}{1-f_{jj}}$.

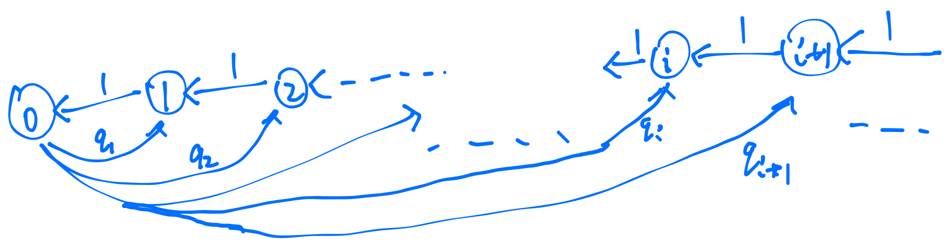
$$\sum_{n\geq 0} p_{jj}^{(n)} = 1 + \sum_{n\geq 1} p_{jj}^{(n)} = \frac{1}{1-f_{jj}}$$

Since $f_{ij} \leq 1$. we have the inequality.

Statement (2). Let P be an irreducible and positive recurrent Markov chain. For any state $i \in S$, let T_i be the first hitting time of i, i.e., $T_i := \inf\{n \geq 1 : X_n = i\}$. We have

$$\mathbb{E}_i[T_i^2] < +\infty.$$

False. Counterexample: Let q be a probability distribution on $\{1, 2, 3 \cdots\}$



$P_{i(i-1)} = 1$ for $i \geq 1$

and $P_{0i} = q_i$ for $i = 1, 2, \cdots$

Clearly the chain is irreducible. Starting from 0, $T_0 - 1 \sim q$.

Choose q such that $\begin{cases} \sum_{i \geq 1} i q_i < +\infty. \\ \sum_{i \geq 1} i^2 q_i = +\infty. \end{cases}$

(Such a q exists, e.g. $q_i = \frac{c}{i^3}$ for $c = \left(\sum_{i=1}^{+\infty} 1/i^3\right)^{-1}$).

We have

$$\mathbb{E}_0[T_0] = \sum_{i \geq 1} (i+1) q_i < +\infty.$$

$$\mathbb{E}_0[T_0^2] = \sum_{i \geq 1} (i+1)^2 q_i = +\infty.$$

Statement (3). Let P be an irreducible Markov chain, and suppose that a stationary distribution π exists. Let μ be a stationary measure of P, with $\mu_i > 0$ for each $i \in S$. Then for $c := \sum_{i \in S} \mu_i$ we have $c < +\infty$, and $\mu = c\pi$.

True. We first show that c is finite.
For any state i, by definition, we have that

$$\forall n, \qquad \mu_i = \sum_{j \in S} \mu_j P_{ji}^{(n)} = \sum_{j \in S} \mu_j \cdot \left(\frac{1}{n} \sum_{t=1}^{n} P_{ji}^{(t)} \right)$$

So for any finite subset S' of S, we have the lower bound

$$\mu_i \geq \sum_{j \in S'} \mu_j \cdot \left(\frac{1}{n} \sum_{t=1}^{n} P_{ji}^{(t)} \right) \xrightarrow[(\text{Taking } n \to +\infty)]{} \left(\sum_{j \in S'} \mu_j \right) \cdot \pi_i$$

The limit argument follows from the average convergence theorem. Note that the inequality holds true for any finite subset S'. So we have

$$c = \sum_{j \in S} \mu_j = \sup_{\substack{S' \subseteq S \\ S' \text{ finite}}} \sum_{j \in S'} \mu_j \leq \frac{\mu_i}{\pi_i} < +\infty \quad (\forall i \in S).$$

Then we can perform the M-test, and note that

$$\sum_{j \in S} \mu_j \cdot \left(\sup_{n \geq 1} \frac{1}{n} \sum_{t=1}^{n} P_{ji}^{(t)} \right) \leq \sum_{j \in S} \mu_j < +\infty.$$

Therefore, we can change the order of limit and the infinite sum, and arrive at the conclusion

$$\begin{aligned} \mu_i &= \lim_{n \to +\infty} \sum_{j \in S} \mu_j \cdot \frac{1}{n} \sum_{t=1}^{n} P_{ji}^{(t)} \\ &= \sum_{j \in S} \mu_j \cdot \lim_{n \to +\infty} \left(\frac{1}{n} \sum_{t=1}^{n} P_{ji}^{(t)} \right) \\ &= \left(\sum_{j \in S} \mu_j \right) \pi_i \\ &= c\, \pi_i. \end{aligned}$$

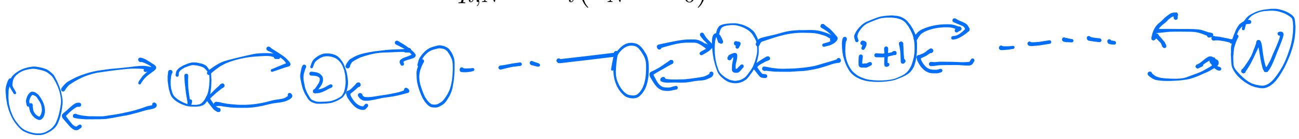
Question 4. [18 pts] Consider a Markov chain on the state space $S = \{0, 1, 2, \cdots\}$. For any $i \geq 1$, we define the transition from the state i as

$$p_{i,i+1} = \frac{i+2}{2i+2}, \quad \text{and} \quad p_{i,i-1} = \frac{i}{2i+2},$$

and $p_{ij} = 0$ for $j \notin \{i-1, i+1\}$. We further let $p_{01} = 1$ and $p_{0j} = 0$ for $j \neq 1$. Apparently this Markov chain is irreducible.

(1) [12 pts]. Define the hitting times $T_j := \inf\{n \geq 1 : X_n = j\}$ for $j \in S$. For integer pairs i, N such that $0 < i < N$, derive a formula for the following probability

$$q_{i,N} := \mathbb{P}_i(T_N < T_0).$$



This is a variant of the gambler's ruin problem, with the only difference being the inhomogeneous winning probabilities at different states. But we can apply the same method using f-expansion.

$$q_{i,N} = \frac{i+2}{2i+2} q_{i+1,N} + \frac{i}{2i+2} q_{i-1,N} \qquad \text{for } i = 1, 2, \cdots, N-1$$

And we set the boundary conditions

$$q_{N,N} = 1, \qquad q_{0,N} = 0.$$

Note that by our recursive formula, (i + 1) * q_{i, N} forms an arithmetic progression. In particular, we note that

$$(i+1)\, q_{i,N} = \frac{1}{2}\left[(i+2)\, q_{i+1,N} + i \cdot q_{i-1,N}\right] \qquad \text{for } i = 1, 2, \cdots, N-1.$$

Solving the arithmetic progression, we have

$$q_{i,N} = \frac{1}{i+1} \cdot \frac{i}{N} \cdot (N+1) = \frac{i(N+1)}{(i+1) \cdot N}.$$

(2) [6 pts]. Conclude that the Markov chain is transient.

Applying the conclusion from the first part with i = 1, we have

$$\mathbb{P}_1(T_N < T_0) = \frac{N+1}{2N} \quad \text{for any} \quad N \geq 2.$$

Note that the Markov chain from the state 1 takes at least (N - 1) steps to visit the state N. So we have

$$\mathbb{P}_1(T_0 \geq N) \geq \mathbb{P}_1(T_0 > T_N) = \frac{N+1}{2N}.$$

The inequality holds true for any N. So we can conclude that

$$\mathbb{P}_1(T_0 = +\infty) \geq \frac{1}{2}.$$

Clearly the chain is irreducible. So we can conclude transience by recurrence equivalence theorem.