STA447/2006: Midterm Exam #2

Instructor: Wenlong Mou

March 22nd, 2024

Student name: ________________

Student ID: ________________

Student signature: ________________

This exam contains 10 pages.

Total marks: 100 pts

Time Allowed: 105 minutes

Question 1. [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

T (1) For a discrete-time stochastic process $(X_n)_{n=0,1,2,\cdots}$, if T_1 and T_2 are both stopping times, then $T_1 \cdot T_2$ is a stopping time.

F (2) For a Brownian motion $(B_t)_{t\geq 0}$, if T_1 and T_2 are both stopping times, then $T_1 \cdot T_2$ is a stopping time.

F (3) Let $(X_n)_{n=0,1,2\cdots}$ be a discrete-time martingale, for any stopping time T such that $\mathbb{P}(T < +\infty) = 1$, we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

F (4) Let $(X_n)_{n=0,1,2\cdots}$ be a discrete-time martingale, if we have

$$\lim_{K\to+\infty} \mathbb{E}[|X_n|\mathbf{1}_{|X_n|\geq K}] = 0, \quad \text{for any } n \geq 0$$

then the process $(X_n)_{n\geq 0}$ is uniformly integrable.

F (5) Let $(X_n)_{n=0,1,2\cdots}$ be a discrete-time irreducible and aperiodic Markov chain on a countably infinite state space $S \subseteq \mathbb{R}$. If $(X_n)_{n\geq 0}$ is also a non-negative martingale that converges some X_∞ almost surely, then the Markov chain $(X_n)_{n\geq 0}$ has a stationary distribution π, and $X_\infty \sim \pi$.

T (6) If $(X_t)_{t\geq 0}$ is a martingale and T is a stopping time, then $(X_{t\wedge T})_{t\geq 0}$ is also a martingale.

T (7) Let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $T := \inf\{t > 0 : B_t = 0\}$. We have $T = 0$ with probability 1.

F (8) Let $(B_t)_{t\geq 0}$ be a Brownian motion and let f be a continuously differentiable function, we have $\mathbb{E}[f(B_t)] = \mathbb{E}\big[\int_0^t f'(B_s)dB_s\big]$.

T (9) Let $(X_t)_{t\geq 0}$ be a two-dimensional standard Brownian motion. With probability 1, for any $t > 0$, there exists $t' > t$, such that $\left\| X_{t'} - \begin{bmatrix} 100 \\ 100 \end{bmatrix} \right\|_2 \leq 10^{-2}$.

F (10) If $(B_t)_{t\geq 0}$ is a Brownian motion, then $\int_0^t B_s^2 dB_s$ is normally distributed.

Question 2. [9 pts] Give an example of a martingale $(X_t)_{t\geq 0}$ such that

$$\sup_{t\geq 0} \mathbb{E}\big[|X_t|\big] \leq C \quad \text{for some constant } C < +\infty$$

and $X_t \to X_\infty$ almost surely as $t \to +\infty$, but we have $\mathbb{E}[X_0] \neq \mathbb{E}[X_\infty]$.

Any valid example with a complete justification deserves full points. A valid example with incomplete or problematic justification gets partial credits (50% -- 90%, depending on the nature of the mistake).

Example:

Let $(S_n)_{n\geq 0}$ be a 1-dimensional simple random walk starting from $S_0 = 1$.

Let $\tau := \min\{t \geq 0 : S_t = 0\}$, which is a stopping time.

$(S_n)_{n\geq 0}$ is martingale $\Rightarrow$ $(X_n := S_{n\wedge\tau})_{n\geq 0}$ is a martingale.

$X_n \geq 0$, so $\sup_{n\geq 0} \mathbb{E}[|X_n|] = \sup_{n\geq 0} \mathbb{E}[X_n] = \mathbb{E}[X_0] = 1$

By recurrence of SRW, $\mathbb{P}(\tau < +\infty) = 1$,

and therefore $X_n \to 0$ a.s.

So $1 = \mathbb{E}[X_0] \neq \mathbb{E}[X_\infty] = 0$.

Question 3. [20 pts] Consider a box N balls with two colours (red and blue). At each round, we draw with replacement from the box for N times independently, and replace the balls in the box with the composition of these random draws. Let X_t be the number of red balls in the t-th round, the process is defined by

$$X_{t+1} = \sum_{j=1}^{N} Z_{t+1,j}, \quad \text{where } Z_{t+1,j}|X_t \overset{\text{i.i.d.}}{\sim} \text{Ber}(X_t/N).$$

Let $X_0 = x \in \{1, 2, \cdots, N-1\}$ fixed

(1) [10 pts]. Show that $(X_t)_{t\geq 0}$ is a martingale, and compute the probability

$$\mathbb{P}\Big(\lim_{t\to+\infty} X_t = 0\Big).$$

Martingale: $\mathbb{E}[X_{n+1} | \mathcal{F}_n] = \sum_{j=1}^{N} \mathbb{E}[Z_{n+1,j} | \mathcal{F}_n] = N \cdot \frac{X_n}{N} = X_n.$

$(X_n)_{n\geq 0}$ is a bounded martingale, so there exists X_∞ s.t. $X_n \to X_\infty$ a.s.

By definition, X_n takes integer value. If X_n stops fluctuating, it must be staying at 0 or N.

$(X_n)_{n\geq 0}$ uniformly bounded $\Rightarrow$ uniformly integrable

Therefore, $x = \mathbb{E}[X_0] = \mathbb{E}[X_\infty] = N \cdot \mathbb{P}(X_\infty = N) + 0 \cdot \mathbb{P}(X_\infty = 0)$

So $\mathbb{P}(X_\infty = 0) = 1 - \frac{x}{N}.$

A correct answer without rigorous justification gets 9 point.

An argument of martingale convergence and/or uniform integrability without a correct answer gets 6 points.

Only showing the martingale property gets 4 points.

(2) [10 pts]. Show that for any $t \geq 0$ and $x \in \{1, 2, \cdots, N-1\}$,

$$\mathbb{P}_x\big(1 \leq X_t \leq N-1\big) \leq x(N-x) \cdot \Big(1 - \frac{1}{N}\Big)^t.$$

[Hint: study the process $H_t := X_t(N - X_t)$.]

$$\mathbb{E}\left[H_{t+1} \,\middle|\, \mathcal{F}_t\right] = N \cdot X_t - X_t^2 - N \cdot \frac{X_t}{N}\left(1 - \frac{X_t}{N}\right)$$

$$= \left(1 - \frac{1}{N}\right) X_t (N - X_t) = \left(1 - \frac{1}{N}\right) H_t$$

So $\quad \mathbb{E}[H_t] = \left(1 - \frac{1}{N}\right)^t H_0 = \left(1 - \frac{1}{N}\right)^t x(N-x)$

By Markov inequality,

$$\mathbb{P}\left(1 \leq X_t \leq N-1\right) = \mathbb{P}\left(H_t \geq 1\right) \leq \mathbb{E}[H_t] = \left(1 - \frac{1}{N}\right)^t x(N-x).$$

7 points if the solution obtains the recursive relation for H_t correctly without establishing the final bound or applying Markov inequality.

Question 4. [25 pts] Let $(B_t)_{t\geq 0}$ be a Brownian motion.

(1) [5 pts]. Let λ be a deterministic constant. If the process $M_t := \exp(\lambda B_t) - 1 - \int_0^t f(B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

$$f(x) = \frac{\lambda^2}{2} e^{\lambda x}$$

$$M_t = \int_0^t \lambda e^{\lambda B_s} dB_s .$$

3 pts if only one of them is computed correctly.

(2) [10 pts]. Compute the quantity [Hint: reflection principle]

$$\mathbb{P}\Big\{\max_{0\le t\le 1} B_t \ge 1, \text{ and } B_1 \le 0\Big\}.$$

(It is fine to express your solution in the form of an integral. You don't need to give a numerical answer.)

Let $\tau := \inf\{t \geq 0, B_t = 1\}$

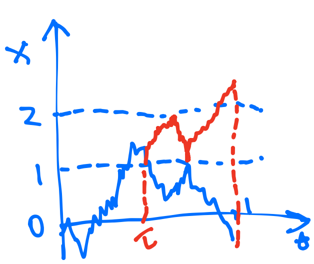
$$\mathbb{P}\Big(\max_{0\le t\le 1} B_t \ge 1, B_1 \le 0\Big)$$

$$= \mathbb{P}(\tau \le 1, B_1 \le 0)$$

$$= \mathbb{P}(\tau \le 1) \cdot \mathbb{P}(B_1 \le 0 \mid \tau \le 1).$$

By strong Markov property and symmetry of Brownian motion

$$\mathbb{P}(B_1 \le 0 \mid \tau \le 1) = \mathbb{P}(B_1 \geq 2 \mid \tau \le 1)$$



So $\mathbb{P}\Big(\max_{0\le t\le 1} B_t \ge 1, B_1 \le 0\Big)$

$$= \mathbb{P}(\tau \le 1) \cdot \mathbb{P}(B_1 \geq 2 \mid \tau \le 1)$$

$$= \mathbb{P}(B_1 \geq 2)$$

$$= \int_2^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Full points if doing the reflection correctly but calculating the integral wrong, or writing in the form of normal cdf, etc.

3 points if the solution assumes independence of the max and B_1, and use reflection principle to compute the probability of max larger than 1, and take the product.

(3) [10 pts]. Let $\tau := \inf\{t \geq 0 : |B_t| \geq 1\}$. Compute var($\tau$).
[Hint: construct a martingale from the process $(B_t^4 - 6tB_t^2)_{t\geq 0}$]

$Z_t := B_t^4 - 6tB_t^2$

By Itô,

$$dZ_t = \left(4B_t^3\, dB_t + 6B_t^2 dt\right) - \left(6B_t^2 dt + 12tB_t\, dB_t + 6t\,dt\right)$$

$$= \left(4B_t^3 - 12tB_t\right) dB_t - 6t\,dt$$

So $\quad Z_t - \int_0^t 6s\,ds = Z_t - 3t^2 = \int_0^t \left(4B_s^3 - 12sB_s\right) dB_s$

is a martingale.

(Exponentially decaying tail of τ has been established in class)

By OST: $\quad \mathbb{E}\left[Z_\tau - 3\tau^2\right] = \mathbb{E}\left[Z_0 - 3\cdot 0^2\right] = 0$

$$\mathbb{E}\left[Z_\tau - 3\tau^2\right] = 1 - 6\,\mathbb{E}[\tau] + 3\,\mathbb{E}[\tau^2]$$

$\mathbb{E}[\tau] = 1$ by applying OST to the martingale $(B_t^2 - t)_{t\geq 0}$
(Already shown in class/text)

So $\quad \mathbb{E}[\tau^2] = \frac{5}{3}$, and $\quad \mathrm{var}(\tau) = \frac{2}{3}$.

9 points for applying the workflow correctly but calculating the final answer wrong.
6 points for successfully constructing the martingale.
Full points if showing the correct answer using any other method.

Question 5. [16 pts] Prove the following statements.

(1) [8 pts]. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Suppose that the process $(X_t)_{t\geq 0}$ starts from $X_0 = x$, satisfying that

$$dX_t = -X_t dt + dB_t.$$

Prove that X_t follows a normal distribution (for any $t > 0$), and compute its mean and variance.

Let $Z_t = e^t X_t$, we have

$$dZ_t = e^t dX_t + e^t X_t dt = -e^t X_t dt + e^t dB_t + e^t X_t dt$$

$$= e^t dB_t$$

$$Z_t = x + \int_0^t e^s dB_s$$

Applying definition of stochastic integration, Z_t is the limit of a sequence of normal random variables, with shared mean of converging variance. Its normality can be established using cdf or characteristic functions (see additional exercises of week 9/10 for details).

$$\mathbb{E}[Z_t] = \mathbb{E}[Z_0] = x.$$

$$\mathrm{var}(Z_t) = \mathrm{var}\left(\int_0^t e^s dB_s\right) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1).$$

So $\quad \mathbb{E}[X_t] = e^{-t} x$

$$\mathrm{var}(X_t) = \frac{1}{2}(1 - e^{-2t}).$$

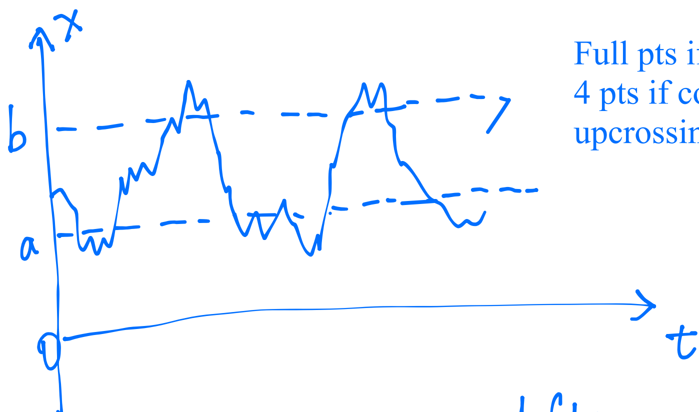
4 points if computing one of the mean/variance correctly.
7 points if computing both correctly without showing normality.
5 points if showing normality correctly.

(2) [8 pts]. Let $(X_n)_{n\geq 0}$ be a discrete-time stochastic process satisfying $\mathbb{E}[|X_n|] < +\infty$ and $\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n$ for any n (such a process is called a super-martingale). Suppose that $X_n \geq 0$ almost surely for each n. Prove that

$$\lim_{n\to+\infty} X_n = X_\infty \quad \text{almost surely,}$$

for some random variable X_∞.

Up-crossing arguments from the martingale convergence theorem.



Full pts if showing the upcrossing inequality correctly.
4 pts if constructing the relevant beeting process and upcrossing counts.

For $0 < a < b < +\infty$, define

$U_n([a,b]) :=$ # of upcrossing $[a,b]$ up to time n.

Construct the betting process Z_n as in class

$$Z_n = \sum_{t=1}^{n} B_t (X_t - X_{t-1})$$

where B_t changes from 0 to 1 if $X_{t-1} \leq a$

from 1 to 0 if $X_{t-1} \geq b$.

$$\mathbb{E}[Z_{t+1} | \mathcal{F}_t] = Z_t + B_{t+1} \cdot \mathbb{E}[X_{t+1} - X_t | \mathcal{F}_t] \leq Z_t.$$

So for any n,

$$0 \geq \mathbb{E}[Z_n] - \mathbb{E}[Z_0] \geq (b-a)\,\mathbb{E}[U_n([a,b])] + \mathbb{E}[X_n - b - X_0]$$

So $\mathbb{E}[U_n([a,b])] \leq \frac{b + \mathbb{E}[X_0]}{b-a}$, apply the same arguments as before.