STA447/2006: Final Exam

Instructor: Wenlong Mou

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Student name: ________________

Student ID: ________________

Student signature: ________________

This exam contains 12 pages.

Total marks: 100 pts

Time Allowed: 180 minutes

Question 1. [20 points, 2.5 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

(1) Let P be an irreducible and transient Markov chain. For any state pair $i, j \in S$, we have $\sum_{n \geq 0} p_{ij}^{(n)} < +\infty$. **Answer:** T

(2) Let P be a reducible and aperiodic Markov chain on a finite state space S. Let i be a transient state. Then there exists a stationary distribution π of P, such that $\lim_{n \to +\infty} p_{ij}^{(n)} = \pi_j$, for each $j \in S$. **Answer:** T

(3) Let i, j be a pair of states in a discrete-time discrete-space Markov chain. If $f_{ij} = 1$, then the chain starting from i will visit j infinitely often. **Answer:** F

(4) Let $(X_k)_{k=0,1,2,\cdots}$ be a real-valued stochastic process. Define

$$T := \inf \Big\{ t \geq 5 : X_t = \max_{0 \leq k \leq t} X_k \Big\}.$$

We have that T is a stopping time. **Answer:** T

(5) Let $(X_k)_{k=0,1,2,\cdots}$ be a martingale that converges to X_∞. Suppose that

$$\mathbb{E}\big[X_k^2\big] < +\infty, \quad \text{for any } k = 0, 1, 2, \cdots$$

Then we have $\mathbb{E}\big[X_\infty\big] = \mathbb{E}\big[X_0\big]$. **Answer:** F

(6) Let $(B(t))_{t \geq 0}$ be a standard Brownian motion. Define $t_{n,i} := i/n$ for $i = 0, 1, 2, \cdots, n$, we have

$$\int_0^1 B(s) ds = \lim_{n \to +\infty} \sum_{i=0}^{n-1} B\Big(\frac{t_{i,n} + t_{i+1,n}}{2}\Big) \cdot \big(t_{i+1,n} - t_{i,n}\big).$$

Answer: T

(7) Let $(B(t))_{t \geq 0}$ be a standard Brownian motion, and define the set $S := \big\{ t \in [0, 1] : B(t) = 1 \big\}$. We have that $\mathbb{P}(|S| = 1) > 0$. **Answer:** F

(8) Let $(N(t))_{t \geq 0}$ be a Poisson process with intensity $\lambda > 0$. The random variables $X = N(3) - N(2)$ and $Y = N(1) + N(4) - N(3)$ are independent. **Answer:** T

Question 2. [14 points] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}.$$

(1) [7 pts]. Compute f_{14}.

Answer: By f-expansion

$$f_{14} = p_{11}f_{14} + p_{12}f_{24},$$
$$f_{24} = p_{24} + p_{21}f_{14}$$

Solving the equation, we get $f_{14} = f_{24} = \frac{2}{3}$.

Rubrics: You will get 6 points if you write the f-expansion formulae correctly with an incorrect calculation for the final answer.

(2) [7 pts]. Find the set of all the stationary distributions of the Markov chain.

Answer: 1,2 are transient states. The stationary distribution puts zero mass on them. 3 is an absorbing state, and the states 4,5 communicate with each other.

The set of stationary distributions is from the linear combination of stationary distributions on all the communicating subsets.

Stationary distribution on 3:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Stationary distribution on $4, 5$:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{3}{5} & \frac{2}{5} \end{bmatrix}.$$

So the set of stationary distributions is

$$\left\{ \begin{bmatrix} 0 & 0 & 1-\lambda & \frac{3}{5}\lambda & \frac{2}{5}\lambda \end{bmatrix} : \lambda \in [0,1] \right\}.$$

Rubrics:

You'll get 5 points if you only get two (extremal points) without realizing their convex combinations are all stationary distributions.

You'll get 3 points if you only find one stationary distribution.

You'll lose 1 point if you get the calculation wrong but the answer is correct otherwise.

You'll get full points if you get the answer correct without justification.

Question 3. [12 points] Consider a Markov chain on the state space $S = \{0, 1, 2, \cdots\}$. For any $i \geq 1$, we define the transition from the state i as

$$p_{i,i+1} = \frac{i}{2i+2}, \quad \text{and} \quad p_{i,i-1} = \frac{i+2}{2i+2},$$

and $p_{i,j} = 0$ for $j \notin \{i-1, i+1\}$. We further let $p_{0,0} = 1/2$, $p_{0,1} = 1/2$ and $p_{0j} = 0$ for $j > 1$. Find the stationary distribution π of the chain, and show that $\lim_{n\to+\infty} p_{ij}^{(n)} = \pi_j$ for each $i, j \in S$

Answer: The chain is reversible with respect to the stationary distribution π. Reversibility condition requires that

$$\frac{i}{2i+2}\pi_i = \frac{i+3}{2i+4}\pi_{i+1}, \quad \text{for } i = 1, 2, \cdots$$

which means

$$3\pi_1 = \cdots = (i-1)(i+1)\pi_{i-1} = i(i+2)\pi_i = (i+1)(i+3)\pi_{i+1}.$$

So we have

$$\pi_i = \frac{3\pi_1}{i(i+2)}, \quad \text{for } i = 1, 2, \cdots$$

For the state 0, we have $\pi_0 p_{0,1} = \pi_1 p_{1,0}$ and therefore $\pi_0 = \frac{3}{2}\pi_1$.

In order to make sure that the stationary distribution sum up to one, we require

$$1 = \frac{3}{2}\pi_1 + \sum_{i=1}^{+\infty} \frac{3\pi_1}{i(i+2)} = \pi_1 \cdot \left\{\frac{3}{2} + \frac{3}{2}\sum_{i=1}^{+\infty}\left(\frac{1}{i} - \frac{1}{i+2}\right)\right\} = \frac{3}{2}\pi_1 \cdot \left\{1 + 1 + \frac{1}{2}\right\}$$

So we have $\pi_1 = \frac{4}{15}$. The stationary distribution is

$$\pi_i = \begin{cases} \dfrac{2}{5} & i = 0, \\ \dfrac{4}{5i(i+2)} & i \geq 1. \end{cases}$$

Convergence follows from the Markov chain convergence theorem, as the chain is irreducible, aperiodic ($p_{0,0} > 0$), and has a stationary distribution.

Rubrics: You'll get 7 points if you realize the chain is reversible and get the correct recursive formula for π_i. (If you get a recursive formula but had some calculation errors so that it's incorrect, you'll get 5 points for this)
2 more points for correctly solving the recursive formula and getting $\pi_i = \frac{3\pi_1}{i(i+2)}$.
2 more points for getting the infinite summation correct and finding the final stationary distribution.
1 more point for verifying the conditions in MC convergence theorem.

Question 4. [27 points] Let $(B_t)_{t\geq 0}$ be standard Brownian motion

(1) [8 pts]. If the process X_t satisfies

$$dX_t = -f(X_t)dt + dB_t,$$

for some smooth function f. Suppose that $M_t := X_t^2 - \int_0^t h(X_s)ds$ is a martingale. Write down the function form of h, and express M_t in the form of an Itô integral.

Answer:

$$d(X_t^2) = 2X_t dX_t + d\langle X\rangle_t = \big(-2X_t f(X_t) + 1\big)dt + 2X_t dB_t.$$

So we have

$$X_t^2 = \int_0^t \big(-2X_s f(X_s) + 1\big)ds + \int_0^t 2X_s dB_s$$

$h(x) = 1 - 2xf(x)$, and $M_t = \int_0^t 2X_s dB_s$.

Rubrics: 4 points for applying the correct Itô formula.
2 more points for computing the quadratic variation part and stochastic integration part correctly.
1 more point for getting the final answer correct.

You'll get full point as long as your final answer is correct.

(2) [8 pts]. Compute the conditional expectation

$$\mathbb{E}\Big[\max_{0\le t\le 1} B_t \mid B_1 = 0\Big].$$

(since the joint distribution has a density, the conditional expectation is well-defined in the classical sense.)

Answer: Let $M = \max_{0\le t\le 1} B_t$. For $x > 0$ and $y < x$, by reflection principle,

$$\mathbb{P}\big(M > x, B_1 < y\big) = \mathbb{P}\big(B_1 > 2x - y\big) = \frac{1}{\sqrt{2\pi}} \int_{2x-y}^{+\infty} e^{-z^2/2} dz.$$

So for any $\varepsilon > 0$, we have

$$\mathbb{P}\big(M > x, B_1 \in (y-\varepsilon, y)\big) = \frac{1}{\sqrt{2\pi}} \int_{2x-y-\varepsilon}^{2x-y} e^{-z^2/2} dz.$$

So we can compute the conditional probability

$$\mathbb{P}\big(M > x \mid B_1 \in (y-\varepsilon, y)\big) = \frac{\int_{2x-y-\varepsilon}^{2x-y} e^{-z^2/2} dz}{\int_{y-\varepsilon}^{y} e^{-z^2/2} dz}.$$

Taking $\varepsilon \to 0$, we have

$$\mathbb{P}\big(M > x \mid B_1 = y\big) = e^{-(2x-y)^2/2} / e^{-y^2/2}.$$

And therefore

$$\mathbb{P}\big(M > x \mid B_1 = 0\big) = e^{-2x^2}.$$

Now we can compute the conditional expectation

$$\mathbb{E}\big[M \mid B_1 = 0\big] = \int_0^{+\infty} e^{-2x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Rubrics:
You'll lose 1 point if you get the idea correct but with some calculation errors.
Full point if getting a correct integral without computing the final answer.
6 points for using reflection principle to derive the formula for $\mathbb{P}\big(M > x, B_1 < y\big)$.
3 more points for using some sorts of Bayes rule to compute the conditional probability or conditional density.
Full points for getting the answer non-rigorously using pictures/intuitions/guesses.

(3) [5 pts]. Let $X_t = B_t - t$ for $t \geq 0$. Show that $M_t = e^{2X_t}$ is a martingale.

Answer: By Itô's formula.

$$dM_t = 2e^{2X_t} dX_t + 2e^{2X_t} dt = 2e^{2X_t} dB_t.$$

So M_t is a stochastic integral, and therefore a martingale.

Rubrics: You'll lose 2 points for any calculation mistake.

Full points if showing the martingale property using other methods (e.g. computing the conditional expectation manually).

(4) [6 pts]. Define the stopping time

$$\tau := \inf\left\{t \geq 0 : |X_t| \geq 1\right\}$$

Use the result in part (3) to compute $\mathbb{P}(X_\tau = 1)$.

Answer: The martingale M_t is bounded up to time τ. So we can apply OST.

$$1 = \mathbb{E}\big[M_0\big] = \mathbb{E}\big[M_\tau\big] = \mathbb{E}\big[e^{2X_\tau}\big] = e^2\mathbb{P}\big(X_\tau = 1\big) + e^{-2}\mathbb{P}\big(X_\tau = -1\big).$$

So we can solve

$$\mathbb{P}\big(X_\tau = 1\big) = \frac{1}{e^2 + 1}.$$

Rubrics:
2 points for using OST to write down the formula.
1 point for justifying the condition for OST.
2 points for getting $e^2\mathbb{P}\big(X_\tau = 1\big) + e^{-2}\mathbb{P}\big(X_\tau = -1\big)$.
1 point for computing the final answer correctly.

Question 5. [10 points] If $(N_1(t))_{t\geq 0}$ and $(N_2(t))_{t\geq 0}$ are two independent Poisson processes, with intensity λ_1 and λ_2 respectively. Given a pair n, t, compute the probability

$$\mathbb{P}\Big(\text{last arrival within } [0,t] \text{ is from } N_2 \mid N_1(t) + N_2(t) = n\Big).$$

Answer: $N = N_1 + N_2$ is a Poisson point process with intensity $\lambda_1 + \lambda_2$. Let τ be the $(n-1)$-th arrival time of N. When $\tau < t$, we study the conditional probability by additionally conditioning on the information up to time τ.

$$\begin{aligned} &\mathbb{P}\Big(\text{last arrival within } [0,t] \text{ is from } N_2 \mid \mathcal{F}_\tau, N(t) = n\Big) \\ &= \mathbb{P}\Big(\text{the arrival within } [\tau,t] \text{ is from } N_2 \mid \mathcal{F}_\tau, \text{only one arrival in } [\tau, t]\Big). \end{aligned}$$

By strong Markov property, this equals the conditional probability that the arrival is from N_2, conditioned on the Poisson process $N = N_1 + N_2$ has only one arrival within time $[0, t-\tau]$. This probability can be computed as

$$e^{-\lambda_2(t-\tau)}\frac{\lambda_2^1}{1!} \cdot e^{-\lambda_1(t-\tau)}\frac{\lambda_1^0}{0!} \Big/ \left(e^{-(\lambda_1+\lambda_2)(t-\tau)}\frac{(\lambda_1+\lambda_2)^1}{1!}\right) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Note that this conditional probability does not depend on $\mathcal{F}_\tau$. So we conclude that

$$\mathbb{P}\Big(\text{last arrival within } [0,t] \text{ is from } N_2 \mid N_1(t) + N_2(t) = n\Big) = \frac{\lambda_2}{\lambda_1+\lambda_2}.$$

Rubrics:
Full points for deriving the correct solution using an alternative approach with a complete justification.
9 points for a correct answer with incomplete justification.
If you tried to attempt this question using brute force and made some partial progress, you will get 1 – 6 points depending on how much progress is made.

Question 6. [17 points] Let and $(X_k)_{k=0,1,2,\cdots}$ be an irreducible Markov chain on the set $S = \{0, 1, 2, \cdots\}$ with transition matrix P. We assume that $|X_{k+1} - X_k| \leq 1$ almost surely for any k. Suppose that there exists a non-negative and strictly increasing function $V : S \to \mathbb{R}_+$ such that $\lim_{x\to+\infty} V(x) = +\infty$.

(1) [10 pts]. If there exists a constant $K < +\infty$, such that the function V satisfies

$$\sum_{j\in S} p_{i,j} V(j) = V(i), \quad \text{for } i > K, \quad \text{and}$$
$$\sum_{j\in S} p_{i,j} V(j) < +\infty, \quad \text{for } i = 0, 1, 2, \cdots, K.$$

Show that the Markov chain $(X_k)_{k=0,1,2,\cdots}$ is recurrent.

Answer: Consider the chain starting from $X_0 = K + 1$. Define the time

$$\tau := \inf\left\{t > 0 : X_t \leq K\right\}.$$

It is easy to see that τ is a stopping time. Define the process

$$M_n = V(X_{n\wedge\tau}), \quad \text{for } n = 0, 1, 2, \cdots$$

By the conditions, we have that M is a non-negative martingale. By martingale convergence theorem, we have

$$\lim_{n\to+\infty} M_n = M_\infty, \quad \text{almost surely,}$$

for some random variable M_∞.

On the other hand, if $\tau = +\infty$, the sequence $\left(X_{n\wedge\tau}\right)_{n\geq 0}$ will either fluctuate infinitely often, or converge to infinity. Since V is strictly increasing and $\lim_{x\to+\infty} V(x) = +\infty$. This will lead to non-convergence of $(M_n)_{n\geq 0}$. Therefore, we have $\tau < +\infty$ almost surely.

Finally, since the Markov chain can move at most one step at a time. The return time of $(K + 1)$ must be smaller than τ, which is also finite.

[Note: the "one step at a time" condition can be completely removed. I just add here to make your life easier.]

Rubrics: Full points for any other alternative proof that is correct.

Partial points for attempting the correct idea but missing some key steps, depending on how close you get.

(2) [7 pts]. Under above setup, if there exist constants $K < +\infty$ and $\varepsilon > 0$, such that the function V satisfies

$$\sum_{j \in S} p_{i,j} V(j) \leq V(i) - \varepsilon, \quad \text{for } i > K, \quad \text{and}$$
$$\sum_{j \in S} p_{i,j} V(j) < +\infty, \quad \text{for } i = 0, 1, 2, \cdots, K.$$

Show that the Markov chain $(X_k)_{k=0,1,2,\cdots}$ has a stationary distribution.

Answer: As before, consider the chain starting from $X_0 = K + 1$. Define the time

$$\tau := \inf \big\{ t > 0 : X_t \leq K \big\}.$$

Define the process

$$M_n := V\big(X_{n \wedge \tau}\big) + \varepsilon(n \wedge \tau).$$

It is easy to see that $\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] \leq M_n$, and therefore

$$\varepsilon \cdot \mathbb{E}\big[n \wedge \tau\big] \leq \mathbb{E}\big[V\big(X_{n \wedge \tau}\big)\big] + \varepsilon \cdot \mathbb{E}\big[n \wedge \tau\big] = \mathbb{E}[M_n] \leq \mathbb{E}[M_0] = V(K+1).$$

So for any n, we have

$$\mathbb{E}\big[n \wedge \tau\big] \leq \frac{V(K+1)}{\varepsilon}.$$

Note that the right-hand-side is independent of n. So we get

$$\mathbb{E}[\tau] \leq \frac{V(K+1)}{\varepsilon} < +\infty.$$

Since the Markov chain can move at most one step at a time. The return time of $(K+1)$ must be smaller than τ, which also has finite expectation. So the Markov chain is positive recurrent, and so the stationary distribution exists.

Rubrics: Full points for any other alternative proof that is correct.
Partial points for attempting the correct idea but missing some key steps, depending on how close you get.
2 points for realizing that the existence of stationary distribution can be established through positive recurrence.