Some additional exercise questions of stochastic calculus

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Throughout the exercise questions, we use $(B_t)_{t\geq 0}$ to denote a standard Brownian motion.

- If the process $M_t := \sin(B_t) - \int_0^t f(B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

Solution: $f(x) = -\frac{1}{2}\sin x$, and

$$M_t = \int_0^t \cos(B_s)dB_s.$$

- If $f : \mathbb{R}_+ \to \mathbb{R}$ is a deterministic continuous function. Show that $\int_0^t f(s)dB_s$ follows a normal distribution, and compute its mean and variance.

Solution: Let $Z = \int_0^t f(s)dB_s$. We have $\mathbb{E}[Z] = 0$ and $\mathrm{var}(Z) = \int_0^t f^2(s)ds$.

Note that

$$Z_n = \sum_{j=0}^{n-1} f\big(jt/n\big) \cdot \Big\{B_{(j+1)t/n} - B_{jt/n}\Big\} \xrightarrow{\mathbb{L}^2} Z.$$

Each Z_n is zero-mean Gaussian, and their variances converges to $\mathrm{var}(Z)$. So we can use the fact that $\mathbb{L}^2$ convergence implies convergence in distribution to show that the CDF of SZ is also Gaussian.