Some additional exercise questions of stochastic calculus

March 27, 2025

Throughout the exercise questions, we use $(B_t)_{t\geq 0}$ to denote a standard Brownian motion.

- If the process $M_t := B_t^4 - 6tB_t^2 - \int_0^t f(s, B_s)ds$ is a martingale. Write down the function form of f, and express M_t in the form of an Itô integral.

Solution: $f(t,x) = 6t$, and

$$M_t = \int_0^t (4B_s^3 - 12sB_s)dB_s.$$

- Based on the previous questions. Compute $\mathbb{E}[T^2]$, where T is the hitting time for $\{-1, 1\}$.

By OST (justification follows from Lawler 8.12)

$$0 = \mathbb{E}[M_T] = 1 - 6\mathbb{E}[T] + 3\mathbb{E}[T^2].$$

From the class, we know that $\mathbb{E}[T] = 1$ (applying OST to $B_t^2 - t$). So we get

$$\mathbb{E}[T^2] = \frac{5}{3}.$$

- Let $Y_t = B_t \cdot \int_0^t B_s dB_s$. Compute dY_t.

Solution: by product rule

$$\begin{aligned} dY_t &= \Big(\int_0^t B_s dB_s\Big) \cdot dB_t + B_t \cdot B_t dB_t + d\langle B, \int_0^\bullet B_s dB_s\rangle_t \\ &= \frac{1}{2}\big(B_t^2 - t\big)dB_t + B_t^2 dB_t + B_t dt. \end{aligned}$$