STA447/2006: Midterm Exam #1

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This exam contains 10 pages.

Total marks: 100 pts

Time Allowed: 110 minutes

Question 1. [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

(1) Let i, j be a pair of states of a Markov chain P. If $f_{ij} < 1$ and j is recurrent, then i is transient. **Answer:** F

(2) There exists an irreducible and transient Markov chain P, such that $f_{ij} < 1$ for any pair $i, j \in S$. **Answer:** T

(3) Let i, j be a pair of states of a Markov chain P. If $i \leftrightarrow j$, and j is null recurrent, then i is also null recurrent. **Answer:** T

(4) If $i \to k$ and $\ell \to j$. When $\sum_{n \geq 0} p_{kl}^{(n)} < +\infty$, we have $\sum_{n \geq 0} p_{ij}^{(n)} < +\infty$. **Answer:** F

(5) There exists a Markov chain that has exactly two stationary distributions. **Answer:** F

(6) Let P be an irreducible and recurrent Markov chain. If $(X_k)_{k \geq 0}$ and $(X'_k)_{k \geq 0}$ are two independent Markov chains following P. Then the joint chain $Y_k = (X_k, X'_k)$ is also irreducible and recurrent. **Answer:** F

(7) Let i, j be a pair of states of an irreducible Markov chain P. If $f_{ij} = f_{ji} = 1$. Then P is recurrent. **Answer:** T

(8) Let i, j be a pair of states of a Markov chain P. If i has period 3 and $i \to j$. Then j also has period 3. **Answer:** F

(9) Let P be an irreducible Markov chain, and let i be a state. If $\mathbb{E}_i[T_i] < +\infty$ (T_i is the first return time to i), then for any j, the limit

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_{ij}^{(n)}$$

exists, and is strictly larger than 0. **Answer:** T

(10) Let P be an irreducible Markov chain on a finite set S. Then for any $i \in S$, we have $\mathbb{E}_i[T_i^2] < +\infty$, where T_i is the first return time to i. **Answer:** T

Question 2. [25 pts] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(1) [5 pts]. Which states are recurrent? Which states are transient? Please explain your reasoning.

Answer: State 4, 5 are recurrent; and state 1,2,3 are transient.

Rubrics: 1pt for each state.

(2) [10 pts]. Compute f_{15} and $\mathbb{E}_1[N(3)]$, where $N(i)$ is the number of visits to the state i. Explain your reasoning. **Answer:** Clearly, we have $f_{55} = 1,\ f_{45} = 0$, since they are absorbing states.

By f-expansion, we have

$$f_{15} = \frac{2}{3} f_{25} + \frac{1}{3} f_{35},$$
$$f_{25} = \frac{1}{3} f_{35} + \frac{1}{3},$$
$$f_{35} = \frac{1}{3} f_{15} + \frac{1}{3}.$$

Solving the system of equations yields $f_{15} = f_{25} = f_{35} = \frac{1}{2}$.

For the state 3, we have $f_{43} = f_{53} = 0$, and by f-expansion

$$f_{13} = \frac{2}{3} f_{23} + \frac{1}{3},$$
$$f_{23} = \frac{1}{3},$$
$$f_{33} = \frac{1}{3} f_{13}.$$

So we have $f_{13} = \frac{5}{9},\ f_{23} = \frac{1}{3},\ f_{33} = \frac{5}{27}$.

Substituting to the formula for expected number of visits, we have

$$\mathbb{E}_1[N(3)] = \frac{f_{13}}{1 - f_{33}} = \frac{15}{22}.$$

Rubrics: Each one of f_{15} and $\mathbb{E}_1[N(3)]$ is worth 5 points. You will lose 1 point (from each) if you get the idea correct but made wrong calculation.

Any other methods with correct answers and complete justification receive full points.

An incomplete answer may get partial credits, depending on the nature of the answer.

(3) [10 pts]. Let $X_0 = 1$, compute the probability that state 2 and state 5 are both visited in the trajectory $(X_k)_{k=0,1,2,\cdots}$.

Answer: Define the event

$$\mathscr{E} := \big\{\text{The chain goes to absorbing state 5}\big\},$$
$$\mathscr{E}' := \big\{\text{The chain goes to absorbing state 5 without going through 2}\big\}.$$

Clearly, we have $\mathscr{E}' \subseteq \mathscr{E}$. From Q(2), we know that $\mathbb{P}_1(\mathscr{E}) = \frac{1}{2}$, and we are interested in $\mathbb{P}(\mathscr{E} \setminus \mathscr{E}') = \mathbb{P}_1(\mathscr{E}) - \mathbb{P}_1(\mathscr{E}')$. So it suffices to compute $\mathbb{P}_1(\mathscr{E}')$.

Define the function $q_i := \mathbb{P}_i(\mathscr{E}')$. We can write down a system of equations for q_i based on next-step transitions, similar to f-expansion.

$$q_1 = \frac{1}{3} q_3,$$
$$q_2 = 0,$$
$$q_3 = \frac{1}{3} q_1 + \frac{1}{3}.$$

Solving it yields $q_1 = \frac{1}{8}$ and $q_3 = \frac{3}{8}$. So we conclude

$$\mathbb{P}_1(\mathscr{E} \setminus \mathscr{E}') = \mathbb{P}_1(\mathscr{E}) - \mathbb{P}_1(\mathscr{E}') = \frac{3}{8}.$$

Rubrics: 5 points for getting the idea of using expansion based on next-step transition.

8 points for getting the entire analysis idea correct without the correct answer.

Any other methods with correct answers and complete justification receive full points.

An incomplete answer may get partial credits, depending on the nature of the answer.

Question 3. [20 pts, 10 pts each] Prove the following statements.

(1). Let P be an irreducible Markov chain in the state space S. If there exist $i, j \in S$, such that $\mathbb{E}_i[T_j] < +\infty$ and $\mathbb{E}_j[T_i] < +\infty$, then P is positive recurrent.

Answer: We note that

$$\mathbb{E}_i[T_i] = \mathbb{E}_i[T_i \mathbf{1}_{T_i \le T_j}] + \mathbb{E}[T_i \mathbf{1}_{T_i > T_j}].$$

Clearly, we have

$$\mathbb{E}_i[T_i \mathbf{1}_{T_i \le T_j}] \le \mathbb{E}_i[T_j \mathbf{1}_{T_i \le T_j}].$$

On the other hand, we note that

$$\mathbb{E}[T_i \mathbf{1}_{T_i > T_j}] = \mathbb{E}[T_j \mathbf{1}_{T_i > T_j}] + \mathbb{E}[(T_i - T_j) \mathbf{1}_{T_i > T_j}]$$

By strong Markov property, after hitting j, the rest of the chain is a fresh new Markov chain starting from j. So we have

$$\mathbb{E}[(T_i - T_j) \mathbf{1}_{T_i > T_j}] = \mathbb{P}(T_i > T_j) \cdot \mathbb{E}_j[T_i] \le \mathbb{E}_j[T_i].$$

Putting them together yields

$$\mathbb{E}_i[T_i] \le \mathbb{E}_i[T_j \mathbf{1}_{T_i \le T_j}] + \mathbb{E}[T_j \mathbf{1}_{T_i > T_j}] + \mathbb{E}_j[T_i] = \mathbb{E}_i[T_j] + \mathbb{E}_j[T_i] < +\infty.$$

Positive recurrence is a communicating class property. Since P is irreducible and i is positive recurrent, the entire chain is therefore positive recurrent.

Rubrics: 7 points for getting the overall idea correct with some missing details.
2 points if only claiming $\mathbb{E}_i[T_i] < +\infty$ without a valid justification.
Incomplete ideas/proofs get some intermediate marks.

(2). There exists a Markov chain P, such that P has a stationary distribution π, but there exists $i \in S$, such that for any state j in the support of π (i.e. a state j with $\pi(j) > 0$), we have

$$i \to j, \quad \text{but} \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_{ij}^{(n)} \neq \pi_j$$

Answer: Here's a construction.

$$P = \begin{bmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

. $\pi = [0, 1/2, 1/2, 0, 0]$ is a stationary distribution, and any states in its support is reachable by 1. However, we have

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} p_{12}^{(n)} = \frac{1}{4} \neq \frac{1}{2}.$$

Rubrics: Any valid construction with justification gets full points.
A correct construction without justification gets 5 points.
Incomplete justification may get intermediate points.

Question 4. [10 pts] Let $S = \{0, 1, 2, \cdots\}$, and consider the following Markov chain

$$P_{0i} = \frac{1}{i(i+1)}, \quad \text{and} \quad P_{i(i-1)} = 1,$$

for every $i = 1, 2, \cdots$. Find a stationary measure of P, and determine whether the Markov chain has a stationary distribution.

Answer: Solve for the stationary condition

$$\pi_i = \pi_{i+1} + \frac{1}{i(i+1)}\pi_0, \quad \text{for } i = 1, 2, \cdots$$
$$\pi_0 = \pi_1.$$

Through a telescope sum, we obtain

$$\pi_1 = \pi_i + \sum_{j=1}^{i-1} \frac{\pi_1}{j(j+1)} = \pi_i + \pi_1 \sum_{j=1}^{i-1} \Big(\frac{1}{j} - \frac{1}{j+1}\Big) = \pi_i + \pi_1\Big(1 - \frac{1}{i}\Big),$$

So we have $\pi_i = \frac{\pi_1}{i}$. A stationary measure does not have to be normalized. So we let $\pi_1 = 1$, and obtain

$$\pi_0 = 1, \quad \text{and} \quad \pi_i = \frac{1}{i}, \quad \text{for } i = 1, 2, \cdots$$

Note that

$$\sum_{i \in S} \pi_i = +\infty.$$

So the chain is null recurrent, and the stationary distribution does not exist.

(Remark: it is also possible to conclude null recurrence by directly computing the expected hitting time).

Rubrics: 7 points for computing the stationary measure correctly without correctly concluding null recurrence.

5 points for proving null recurrence correctly without finding a correct stationary measure.

2 points for writing down the correct stationary equation without solving it.

You'll lose 2 points if you made some minor calculation mistake that does not affect the nature of the result.

Question 5. [15 pts] Let P be a reversible Markov chain over a finite state space S ($|S| < +\infty$). Let π be its stationary distribution. Suppose that P is irreducible and aperiodic. From the class, we know that the Markov chain converges to its stationary distribution. Through this question, we will quantify how fast it converges.

(1) [6 pts]. Show that the matrix P is diagonalizable and all its eigenvalues are real.

Answer: By reversibility, we have

$$\pi_i p_{ij} = \pi_j p_{ji}, \quad \text{for any } i, j \in S.$$

Re-arranging yields

$$\sqrt{\pi_i} p_{ij} / \sqrt{\pi_j} = \sqrt{\pi_j} p_{ji} / \sqrt{\pi_i}.$$

In matrix notation, this can be re-written as

$$\left(\Pi^{1/2} P \Pi^{-1/2}\right)^\top = \Pi^{1/2} P \Pi^{-1/2},$$

where $\Pi = \text{diag}\left(\pi_i\right)_{i \in S}$.

So the matrix $\Pi^{1/2} P \Pi^{-1/2}$ is symmetric, and therefore diagonalizable with real eigenvalues.

Suppose that $\Pi^{1/2} P \Pi^{-1/2} = V D V^{-1}$ for real-diagonal matrix D, we have

$$P = (\Pi^{-1/2} V) D (\Pi^{-1/2} V)^{-1},$$

which is also diagonalizable with real eigenvalues.

Rubrics: 4 points for deriving the matrix $\Pi^{1/2} P \Pi^{-1/2}$ without a complete proof.

2 points for realizing the idea of relating reversibility to symmetric matrices, but without the correct construction.

(2) [9 pts]. Show that there exists a pair of constants $c, \lambda > 0$ depending on P, such that

$$\sum_{j \in S} \left| p_{ij}^{(n)} - \pi_j \right| \leq c e^{-\lambda n},$$

for every $i \in S$.
[Hint: you can use the result of part (1) as given, even if you have not proven it.]

Answer: Since $\pi = \pi P$, we know that π is a left eigen-vector of P, with eigen-value 1. On the other hand, we note that for any function f on the state space, we have

$$\mathbb{E}_{X_0 \sim \pi}\big[\, |(Pf)(X_0)|^2 \,\big] = \mathbb{E}_{X_0 \sim \pi}\big[\mathbb{E}[f(X_1) \mid X_0]^2\big] \leq \mathbb{E}\big[f(X_1)^2\big] = \mathbb{E}\big[f(X_0)^2\big].$$

So we have that

$$\sum_{i \in S} \pi_i (Pf)_i^2 \leq \sum_{i \in S} \pi_i f_i^2, \quad \text{for any function } f,$$

and consequently,

$$\|\Pi^{1/2} P \Pi^{-1/2}\|_{\mathrm{op}}^2 = \sup_f \frac{\sum_{i \in S} \pi_i (Pf)_i^2}{\sum_{i \in S} \pi_i f_i^2} \leq 1.$$

So the eigenvalues of P has absolute value at most 1. By uniqueness of stationary distribution, the eigenvalue 1 cannot have a multiplicity of more than 1. If -1 is an eigen-value with left eigen-vector u, we have $u = -uP$. For any $\varepsilon > 0$, we have

$$(\pi + \varepsilon u)P^n = \pi P^n + \varepsilon u P^n = \pi + (-1)^n \varepsilon u.$$

We can choose $\varepsilon > 0$ small enough and fixed, such that $\pi + \varepsilon u$ is element-wise positive, so that normalizing it yields a probability distribution. This implies that a chain starting from such a distribution will oscillate and cannot converge, which leads to contradiction.

So we conclude that, except for the eigen-value 1 corresponding to the left-eigen-value π, all the eigen-values have absolute value strictly less than 1. We have $P = U^{-1} D U$, with

$$D = \mathrm{diag}(1, \lambda_2, \lambda_3, \cdots, \lambda_{|S|}),$$

For any i, we have

$$\left| e_i^\top P^n e_j - \pi_j \right| = \left| e_i U^{-1} D^n U e_j - e_i U^{-1} \mathrm{diag}(1, 0, 0, \cdots, 0) U e_j \right| \leq \|U\|_{\mathrm{op}} \cdot \|U^{-1}\|_{\mathrm{op}} \cdot \max_{2 \leq k \leq |S|} |\lambda_k|^n,$$

which is exponentially decaying.

Rubrics: 6 points for getting the idea of power iteration and eigen-values bounded by 1, even without a complete justification.

3 points for realizing the key step is to prove eigen-value absolute value bound.

There are several other ways of proving this result, which may not rely on eigen-decomposition. Other proofs or proof ideas will also get credit.