# STA447/2006: Midterm Exam #2

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**This exam contains 11 pages.**

**Total marks: 100 pts**

**Time Allowed: 110 minutes**

**Question 1.** [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

(1) For a discrete-time stochastic process $(X_n)_{n=0,1,2,\cdots}$, if $T_1$ and $T_2$ are both stopping times, then $T_1^2 + 2T_2$ is a stopping time. **Answer:** T

(2) For a discrete-time stochastic process $(X_n)_{n=0,1,2,\cdots}$, if $T_1$ and $T_2$ are both stopping times satisfying $T_1 \geq T_2$, then $2T_1 - T_2$ is a stopping time. **Answer:** T

(3) Let $T := \arg\max_{t\geq 0} X_t$. The random time $T$ is a stopping time. **Answer:** F

(4) If $(Z_i)_{i\geq 0}$ are i.i.d. zero-mean random variables, the following process $(X_n)_{n\geq 0}$ is a martingale

$$X_0 = 0, \quad \text{and for } n \geq 1 \quad X_n := \sum_{i=1}^{n} (Z_i + Z_{i-1}).$$

**Answer:** F

(5) Let $(X_n)_{n\geq 0}$ be a discrete-time martingale. If there exists $C < +\infty$, such that $\mathbb{E}[|X_n|] \leq C$ for each $n$. Then we have the convergence results $X_n \to X_\infty$ a.s., and $\mathbb{E}[X_n] \to \mathbb{E}[X_\infty]$. **Answer:** F

(6) Let $(X_n)_{n=0,1,2\cdots}$ be a discrete-time martingale, if we have

$$\lim_{K\to+\infty} \sup_{n\geq 0} \mathbb{E}\Big[|X_n| \mathbf{1}_{|X_n|\geq K}\Big] = 0, \quad \text{for any } n \geq 0,$$

then the process $(X_n)_{n\geq 0}$ is uniformly integrable. **Answer:** T

(7) Let $(X_n)_{n=0,1,2\cdots}$ be a martingale. If $T$ is a stopping time satisfying $\mathbb{P}(T < +\infty) = 1$, and $\mathbb{E}[|X_T|] < +\infty$. Then we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. **Answer:** F

(8) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. The process $(B_t^3)_{t\geq 0}$ is a continuous-time martingale. **Answer:** F

(9) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Define the random time

$$T := \inf\big\{t > 0 \ : \ B_t = 0\big\}.$$

Then we have $\mathbb{P}(T = 0) = 1$. **Answer:** T

(10) Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. The random variable $X := \int_0^1 B_t dt$ follows a normal distribution. **Answer:** T

**Question 2.** [24 pts] For each of the following stochastic process $(X_t)_{t\geq 0}$ and random time $T$, compute $\mathbb{E}[X_T]$, and *justify your answer*. (If you just give the number without justification, you will only receive a small portion of the credit).

***(1) [8 pts].*** Let $X_0 = 1$ and $X_t = W_t X_{t-1}$ where $W_t$ are i.i.d. random variables, following the distribution

$$W_t = \begin{cases} 2 & \text{with probability } 1/3, \\ \frac{1}{2} & \text{with probability } 2/3, \end{cases}$$

and the random time $T := \inf\{t > 0 : X_t \geq 8\}$.
[Hint: for this process, we may have $\mathbb{P}(T = +\infty) > 0$, and in such a case, we use $X_\infty$ to denote the almost sure limit of the process $(X_t)_{t\geq 0}$.]

**Answer:** Note that $\mathbb{E}[X_t \mid \mathcal{F}_{t-1}] = X_{t-1}\mathbb{E}[W_t] = X_{t-1}$. So the process $(X_t)_{t\geq 0}$ is a martingale.

Consider the process $Y_n := X_{n\wedge T}$. $(Y_n)_{n\geq 0}$ is also a martingale, satisfying $|Y_n| \leq 8$ almost surely. So the process $(Y_n)_{n\geq 0}$ is uniformly integrable.

By martingale convergence theorem, $Y_n \to Y_\infty$ for some limiting random variable $Y_\infty$, and we have $\mathbb{E}[Y_\infty] = \mathbb{E}[Y_0] = 1$. On the other hand, we note that $Y_\infty = X_T$ by definition. So we get $\mathbb{E}[X_T] = 1$.

Remark: This is an example of "OST and martingale convergence" we discussed in Lecture 8 (see page 4 of the notes).

**Rubrics:** 2 points for showing martingale.

3 points for getting the correct answer.

3 points for the correct reasoning steps. (If you just apply OST without discussing how we treat the $T = +\infty$ case, you will not get these 3 points).

If you realize the fact that $X_T$ is either 0 or 8. You will also get 2 points.

Note that there're other ways of proving this result as well (e.g. treating this as a birth-death chain and using Markov chain techniques). You will get full credit if you provide a correct answer using this technique.

***(2) [8 pts].*** Let $(Z_t)_{t=0,1,2,\cdots}$ be 1-dimensional symmetric simple random walk, and

$$X_t = Z_t^4 - 6\sum_{i=0}^{t-1} Z_i^2 - t.$$

and the random time $T := \inf\{t > 0 : |Z_t| \geq 5\}$.

**Answer:** Let $\varepsilon_{t+1} = Z_{t+1} - Z_t$. Note that

$$\begin{aligned}\mathbb{E}\big[X_{t+1} \mid \mathcal{F}_t\big] &= \mathbb{E}\big[(Z_t + \varepsilon_{t+1})^4 \mid \mathcal{F}_t\big] - 6\sum_{i=0}^{t} Z_i^2 - (t+1) \\ &= Z_t^4 + 6Z_t^2 + 1 - 6\sum_{i=0}^{t} Z_i^2 - (t+1) \\ &= Z_t^4 - 6\sum_{i=0}^{t-1} Z_i^2 - t.\end{aligned}$$

So $(X_t)_{t\geq 0}$ is a martingale. We note that

$$\mathbb{E}\big[|X_T|\mathbf{1}_{T\geq n}\big] \leq \mathbb{E}\Big[\big(5^4 + 6\cdot 5^2 T + T\big)\mathbf{1}_{T\geq n}\Big] = \mathbb{E}\Big[\big(625 + 151T\big)\mathbf{1}_{T\geq n}\Big].$$

Before hitting $\pm 5$, $(Z_t)_{t\geq 0}$ is running as a finite-state-space irreducible Markov chain. So there exists $c > 0, \rho < 1$, such that $\mathbb{P}\big(T \geq n\big) \leq c\rho^n$, and consequently,

$$\mathbb{E}\Big[\big(625 + 151T\big)\mathbf{1}_{T\geq n}\Big] \to 0.$$

So we can apply OST to conclude that $\mathbb{E}[X_T] = 0$.

**Rubrics:** 4 points for showing martingale.

4 points for correctly verifying the condition for OST.

You will get some partial credit if you make some progress in one of the two steps but did not get through in the end.

***(3) [8 pts].*** Let $\varepsilon_t$ be i.i.d. random variables with $\mathbb{P}(\varepsilon_t = 1) = 2/3$ and $\mathbb{P}(\varepsilon_t = -1) = 1/3$. Define the process $X_t = \sum_{j=1}^{t} \varepsilon_j$. Consider the auxiliary process $Y_t$ defined as $Y_0 = 0$, and

$$Y_{t+1} := \begin{cases} \widetilde{Y}_{t+1} & \widetilde{Y}_{t+1} \in \{0, 1, \cdots, 19\}, \\ 0 & \widetilde{Y}_{t+1} = 20, \\ 19 & \widetilde{Y}_{t+1} = -1, \end{cases} \qquad \text{where } \widetilde{Y}_{t+1} = Y_t + \varepsilon_{t+1}.$$

(i.e., we arrange the numbers $\{0, 1, 2, \cdots, 19\}$ in a circle, and for each time, $Y$ moves according to the direction of $\varepsilon$).

Define the random time $T := \inf\{t > 0 : Y_t = 0\}$.

**Answer:** The process $(X_t)_{t \geq 0}$ is partial sum of i.i.d. integrable random variables. By Wald's theorem, we have

$$\mathbb{E}[X_T] = \mathbb{E}[\varepsilon_1] \cdot \mathbb{E}[T],$$

as long as $\mathbb{E}[T] < +\infty$.

On the other hand, $T$ is the first time that the Markov chain $(Y_t)_{t \geq 0}$ returns to the starting state 0. Note that this Markov chain is irreducible in a finite state space, therefore positive recurrent. By symmetry, the stationary distribution is

$$\pi = \begin{bmatrix} \frac{1}{20} & \frac{1}{20} & \cdots & \frac{1}{20} \end{bmatrix}.$$

So we have $\mathbb{E}[T] = 1/\pi_0 = 20$, and therefore

$$\mathbb{E}[X_T] = \frac{20}{3}.$$

**Rubrics:** 4 points for invoking Wald's theorem correctly.

4 points for computing the expected hitting time.

If you got all the idea correct but made some calculation mistakes, you'll get 6 points.

**Question 3.** [25 pts] Let $(B_t)_{t\geq 0}$ be a standard Brownian motion.

***(1) [10 pts].*** Compute the following probability

$$\mathbb{P}\Big(\max_{0\leq t\leq 1} B_t \geq 1, B_1 \geq 0\Big).$$

You may express your solution using the notation $\Phi(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz$, i.e., the cumulative distribution function of standard normal distribution.

**Answer:** Define the stopping time

$$T := \inf\Big\{t \geq 0 : B_t = 1\Big\}.$$

Note that

$$\begin{aligned}
&\mathbb{P}\Big(\max_{0\leq t\leq 1} B_t \geq 1, B_1 \geq 0\Big)\\
&= \mathbb{P}\big(T \leq 1, B_1 \geq 0\big)\\
&= \mathbb{P}(T \leq 1) - \mathbb{P}\big(T \leq 1, B_1 < 0\big).
\end{aligned}$$

By reflection principle, we have

$$\begin{aligned}
\mathbb{P}(T \leq 1) &= 2\mathbb{P}(B_1 > 1), \quad \text{and}\\
\mathbb{P}(T \leq 1, B_1 < 0) &= \mathbb{P}(B_1 > 2).
\end{aligned}$$

So we have

$$\mathbb{P}\Big(\max_{0\leq t\leq 1} B_t \geq 1, B_1 \geq 0\Big) = 2\Phi(-1) - \Phi(-2).$$

**Rubrics:** All equivalent expressions will get the full credit.

8 points if the flow of arguments is completely correct but made some calculation mistakes in the final answer.

5 points if one of $\mathbb{P}(T \leq 1)$ and $\mathbb{P}(T \leq 1, B_1 < 0)$ is correctly computed.

3 points if getting the idea of using reflection principle in some way (e.g. reasoning about the hitting time), without finishing the computation.

***(2) [6 pts].*** Let $Y_t = B_t - \frac{1}{2}t$. Show that $M_t = \exp(Y_t)$ is a martingale.

**Answer:** Note that for $s < t$,

$$\mathbb{E}\big[e^{Y_t} \mid \mathcal{F}_s\big] = e^{B_s - t/2} \cdot \mathbb{E}\big[e^{B_t - B_s} \mid \mathcal{F}_s\big] = e^{B_s - t/2} \cdot e^{(t-s)/2} = e^{Y_s},$$

where we used the fact that $B_t - B_s \sim \mathcal{N}(0, t-s)$ independent of $\mathcal{F}_s$, and applied the expression for moment generating function of Gaussian random variables.

Clearly, $\mathbb{E}[e^{Y_t}] < +\infty$ for any $t$. So by definition, $e^{Y_t}$ is a martingale.

**Rubrics:** 3 points for verifying the correct condition for martingales.

3 points for the calculation.

***(3) [9 pts].*** Define the stopping time

$$T := \inf\Big\{t > 0 \ : \ |Y_t| = 1\Big\}.$$

Compute $\mathbb{P}(Y_T = 1)$.
[Hint: you may take the conclusion of part (2) as given, even if you do not know how to prove it. You will get full credit as long as the final numbers are correct.]

**Answer:** Note that $e^{Y_t} \in [e^{-1}, e]$ before time $T$. So the martingale is bounded up to time $T$. Applying OST, we have

$$\mathbb{E}\big[e^{Y_0}\big] = \mathbb{E}\big[e^{Y_T}\big] = e\mathbb{P}(Y_T = 1) + \frac{1}{e}\mathbb{P}(Y_T = -1).$$

On the other hand, $\mathbb{P}(Y_T = 1) + \mathbb{P}(Y_T = -1) = 1$. Solving the equation yields

$$\mathbb{P}(Y_T = 1) = \frac{1}{e+1}.$$

**Rubrics:** 7 points if the flow of arguments is completely correct but some calculation mistake is made.

3 points if applying OST to the right martingale without knowing how to start from there.

**Question 4.** [21pts] Proof-based questions.

***(1) [7pts].*** Let $(X_n)_{n\geq 0}$ be a discrete-time martingale, if there exists a random variable $Z$ such that $\mathbb{E}[|Z|] < +\infty$, and $|X_n| \leq Z$ for each $n$. Show that there exists a limiting random variable $X_\infty$, such that $\mathbb{E}[|X_n - X_\infty|] \to 0$.

**Answer:** Since $\mathbb{E}[|X_n|] \leq \mathbb{E}[|Z|] < +\infty$, by martingale convergence theorem, there exists $X_\infty$, such that $X_n \to X_\infty$ almost surely. By Fatou's lemma, we have

$$\mathbb{E}[|X_\infty|] \leq \lim \inf_{n\to+\infty} \mathbb{E}[|X_n|] \leq \mathbb{E}[|Z|] < +\infty.$$

Note that $|X_n - X_\infty| \leq |Z| + |X_\infty|$, where the right-hand-side has finite expectation. By dominating convergence theorem, we have

$$\lim_{n\to+\infty} \mathbb{E}\big[|X_n - X_\infty|\big] = \mathbb{E}\big[\lim_{n\to+\infty} |X_n - X_\infty|\big] = 0.$$

**Rubrics:** Any other valid proof (e.g. by showing uniform integrability of the martingale) receives full credit.

6 points if using DCT directly without justifying the integrability of $X_\infty$.

2 points if showing a.s. convergence using martingale convergence theorem, without showing $L^1$ convergence.

***(2) [7pts].*** Construct a discrete-time martingale $(X_n)_{n\geq 0}$ and a stopping time $T$ such that $\mathbb{E}[T] < +\infty$ and $\mathbb{E}[|X_T|] < +\infty$, but $\mathbb{E}[X_T] \neq \mathbb{E}[X_0]$. Justify your answer.

**Answer:** Let $Z_1, Z_2, \cdots$ be an infinite sequence of i.i.d. random variables, satisfying

$$Z_i = \begin{cases} 2 & \text{with probability } 1/2, \\ 0 & \text{with probability } 1/2. \end{cases}$$

Let $X_0 = 1$, and $X_n = X_{n-1} Z_n$ for $n \geq 1$, and define the stopping time

$$T := \inf\{t \geq 0 : X_t = 0\}.$$

Note that for every $n > 0$, we have

$$\mathbb{P}(T > n) = \mathbb{P}(Z_1 = Z_2 = \cdots = Z_n = 2) = 2^{-n}.$$

So we have $\mathbb{P}(T < +\infty) = 1$ and $\mathbb{E}[T] = 2 < +\infty$.

On the other hand, we have $X_T = 0$ almost surely, and $\mathbb{E}[X_0] = 1$. So $0 = \mathbb{E}[X_T] \neq \mathbb{E}[X_0] = 1$.

**Rubrics:** Any other valid example with complete justification receives full credit.

A correct construction with incomplete justification receives 5 points.

Any incorrect construction that is not sum of i.i.d. random variables receives 2 point. (At least from Wald's theorem you should know what kind of process to avoid).

***(3) [7pts].*** If $(X_n)_{n=0,1,2,\cdots}$ is a non-negative martingale. Define

$$Y_n := \max_{0 \le i \le n} X_i.$$

Prove that there exists $C > 0$ such that $\mathbb{E}\big[\sqrt{Y_n}\big] \le C$ for any $n$.

**Answer:** By Doob's maximal inequality, for any $z > 0$, we have

$$\mathbb{P}(Y_n \ge z) \le \frac{\mathbb{E}[X_n]}{z} \le \frac{\mathbb{E}[X_0]}{z}.$$

Note that

$$\begin{aligned}
\mathbb{E}\big[\sqrt{Y_n}\big] &= \int_0^{+\infty} \mathbb{P}\big(\sqrt{Y_n} \ge t\big)\, dt \\
&= \int_0^{\sqrt{\mathbb{E}[X_0]}} \mathbb{P}\big(\sqrt{Y_n} \ge t\big)\, dt + \int_{\sqrt{\mathbb{E}[X_0]}}^{+\infty} \mathbb{P}\big(\sqrt{Y_n} \ge t\big)\, dt \\
&\le \int_0^{\sqrt{\mathbb{E}[X_0]}} dt + \int_{\sqrt{\mathbb{E}[X_0]}}^{+\infty} \frac{\mathbb{E}[X_0]}{t^2}\, dt \\
&= 2\sqrt{\mathbb{E}[X_0]},
\end{aligned}$$

which is a uniform upper bound independent of $n$.

**Rubrics:** Any other valid proof receives full credit, no matter what final bound you get.

3 points for invoking Doob's maximal inequality correctly.

2 points for writing the integral form of expectation (or something similar).