STA447/2006: Final Exam

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This exam contains 14 pages.

Total marks: 100 pts

Time Allowed: 180 minutes

Question 1. [20 points, 2.5 points each] Mark each statement with T (true) or F (false). No justification required.

(1) Let P be an irreducible and transient discrete-time Markov chain. For any state pair $i, j \in S$, we have $f_{ij} < 1$. **Answer:** F

(2) Let i, j, k, ℓ be states in a discrete-time discrete-space Markov chain. If $i \to j$ and $k \to \ell$. Suppose that $\sum_{n \geq 0} p_{i\ell}^{(n)} < +\infty$. Then $\sum_{n \geq 0} p_{jk}^{(n)} < +\infty$. **Answer:** T

(3) Let P be an irreducible Markov chain. If P is positive recurrent, then for any i, j pair, we have $\mathbb{E}_i[T_j] < +\infty$, where T_j denotes hitting time of j. **Answer:** T

(4) Let $(X_k)_{k=0,1,2,\cdots}$ be symmetric simple random walk. Let us define

$$T := \sup \Big\{ t \in [0, 100] : X_t = 0 \Big\}.$$

We have that T is a stopping time. **Answer:** F

(5) Let $(X_k)_{k=0,1,2,\cdots}$ be a martingale. Suppose that we have $X_k \geq -20$ almost surely, for any $k \geq 0$. Then there exists a limiting random variable X_∞, such that $\mathbb{P}(X_n \to X_\infty) = 1$. **Answer:** T

(6) Let $(B(t))_{t \geq 0}$ be a standard Brownian motion, we have

$$\int_0^t dB(s) = B(t).$$

Answer: T

(7) For $i = 1, 2, 3$, let $(B^{(i)}(t))_{t \geq 0}$ be independent standard Brownian motions. Define the random time

$$T := \inf \Big\{ t \geq 0 : B^{(i)}(t) \in [10, 11] \quad \text{for } i = 1, 2, 3 \Big\}.$$

Then we have $\mathbb{P}(T < +\infty) = 1$. **Answer:** F

(8) Let $(N(t))_{t \geq 0}$ be a Poisson process with intensity $\lambda = 1$. Let $X(t) = N(t)^2 - (t + t^2)$ for any $t > 0$. The process $(X(t))_{t \geq 0}$ is a martingale. **Answer:** F

Question 2. [14 points] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix}.$$

For $i \in S$, define the hitting time

$$T_i := \inf\{t \geq 1 : X_t = i\}.$$

(1) [7 pts]. Compute $\mathbb{E}[T_1 \mid X_0 = 1]$.

Answer: By symmetry, a stationary distribution is

$$\pi_i = 1/5, \quad \text{for } i \in \{1, \cdots, 5\}.$$

Since the Markov chain is irreducible, the stationary distribution is unique.

By recurrence time theorem, we have

$$\mathbb{E}_1[T_1] = \frac{1}{\pi_1} = 5.$$

Rubrics: 5 points for using (the idea of) recurrence time theorem.

2 points for computing the stationary distribution correctly.

Any other method with correct answer gets full credit.

(An alternative method is to use one-step expansion for $\mathbb{E}_i[T_1]$ and solve a system of equations. Partial progress using this approach will get partial credit, e.g., listing the correct system of equations is worth 5 points).

(2) [7 pts]. Compute $\mathbb{P}\big(T_5 < T_2 \mid X_0 = 1\big)$.

Answer: Let $q_i := \mathbb{P}_i(T_5 < T_2)$, we have the one-step expansion (similar to f-expansion)

$$q_1 = \frac{1}{2} \cdot 0 + \frac{1}{2} q_3$$
$$q_3 = \frac{1}{2} q_4 + \frac{1}{2} \cdot 1$$
$$q_4 = \frac{1}{2} \cdot 1 + \frac{1}{2} q_1$$

Solving the linear system, we get $q_1 = \frac{3}{7}$.

Rubrics: 5 points for listing the correct system of equations.

This problem can also be solved using f-expansion, by considering a modified chain where 2 and 5 become absorbing states. If you get this idea but did not list the correct system of f-expansion equations, you'll get 3 points.

Any other method with the correct answer gets full credit, with partial progress getting partial credit.

Question 3. [13 points] Consider a Markov chain on the state space $S = \{0, 1, 2, \cdots\}$. Let q be a scalar in $(0, 1)$. For any $i \geq 1$, we define the transition from the state i as

$$p_{i,i+1} = q, \quad \text{and} \quad p_{i,i-1} = 1 - q,$$

and $p_{i,j} = 0$ for $j \notin \{i-1, i+1\}$. We further let $p_{0,0} = 1/2$, $p_{0,1} = 1/2$ and $p_{0j} = 0$ for $j > 1$.

(1) [6 pts]. Find a stationary measure of the Markov chain P. (The answer depends on q).

Answer: Let μ be a stationary measure. Detailed balance conditions gives

$$\mu_i q = \mu_{i+1}(1-q) \quad \text{for } i = 1, 2, \cdots$$

Therefore, we have

$$\mu_i = \mu_1 \left(\frac{q}{1-q}\right)^{i-1}.$$

We also have

$$\frac{1}{2}\mu_0 = (1-q)\mu_1.$$

So $\mu_0 = 2(1-q)\mu_1$.

Let $\mu_1 = 1$ without loss of generality, a stationary measure is

$$\mu_0 = 2(1-q), \quad \text{and} \quad \mu_i = \left(\frac{q}{1-q}\right)^{i-1} \quad \text{for } i \geq 1.$$

(Any positive constant multiple of this is also a stationary measure).

Rubrics: 3 points for writing down the detailed balance condition. (If you write down the original definition of stationary measure, it is also worth 3 points).

If you made some calculation mistake in finding μ_0 only, you lose 1 point.

If you made some calculation mistake in finding the general expression of μ_i, you lose 2 points.

(2) [7 pts]. Find the set of values of $q \in (0,1)$ such that the chain is positive recurrent, null recurrent, and transient, respectively. Justify your answer.

Answer: When $q < \frac{1}{2}$, the stationary measure satisfies $\sum_i \mu_i < +\infty$, and the chain is positive recurrent.

Let us consider the case of $q \geq \frac{1}{2}$. Define the quantity

$$p_{i,N} := \mathbb{P}_i(T_0 > T_N).$$

Using the known result on gambler's ruin problem in class, we have

$$p_{i,N} = \begin{cases} \dfrac{i}{N}, & q = \dfrac{1}{2}, \\ \dfrac{1 - (\frac{1-q}{q})^i}{1 - (\frac{1-q}{q})^N}, & q > \dfrac{1}{2}. \end{cases}$$

Fix i and take $N \to +\infty$, we have

$$\mathbb{P}_i\big(\text{never visit } 0\big) = \lim_{N \to +\infty} p_{i,N} = \begin{cases} 0, & q = \dfrac{1}{2}, \\ 1 - (\frac{1-q}{q})^i, & q > \dfrac{1}{2}. \end{cases}$$

Therefore, the chain is null recurrent when $q = \frac{1}{2}$, and transient when $q > \frac{1}{2}$.

Rubrics: 2 points for positive recurrence. 3 points for null recurrence. 2 points for transience.

For each part, a correct answer with incorrect justification gets 1 point.

Question 4. [28 pts] Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(1) [7 pts]. Compute the limit

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} B_{\frac{i+1}{N}}\cdot\left(B_{\frac{i+1}{N}}-B_{\frac{i}{N}}\right).$$

The answer should be in closed form (i.e., no limits or integrals in your final expression).

Answer:

$$\begin{aligned}&\lim_{N\to+\infty}\sum_{i=0}^{N-1} B_{\frac{i+1}{N}}\cdot\left(B_{\frac{i+1}{N}}-B_{\frac{i}{N}}\right)\\ &=\lim_{N\to+\infty}\sum_{i=0}^{N-1} B_{\frac{i}{N}}\cdot\left(B_{\frac{i+1}{N}}-B_{\frac{i}{N}}\right)+\lim_{N\to+\infty}\sum_{i=0}^{N-1}\left(B_{\frac{i+1}{N}}-B_{\frac{i}{N}}\right)^2.\end{aligned}$$

By definition, the first term is an Itô integral, and the second term is a quadratic variation. So we have

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} B_{\frac{i+1}{N}}\cdot\left(B_{\frac{i+1}{N}}-B_{\frac{i}{N}}\right)=\int_0^1 B_t dB_t+\langle B\rangle_1=\frac{1}{2}B_1^2+\frac{1}{2}.$$

Rubrics: Any solution with the correct final answer gets full credit.

You will get 5 points if the overall idea is correct but you made some calculation mistakes.

You will get 3 points if you relate the limit to an Itô's integral and compute the Itô's integral, but fail to get the additional terms correctly.

(2) [7 pts]. Suppose that $(X_t)_{t\geq 0}$ satisfies

$$dX_t = f(X_t)dB_t,$$

for a continuous function f. Suppose that $M_t := e^{X_t} - \int_0^t h(X_s)ds$ is a martingale. Write down the function form of h, and express $M_t - M_0$ in the form of an Itô integral.

Answer: By Itô's formula, we have

$$\begin{aligned} d(e^{X_t}) &= e^{X_t}dX_t + \frac{1}{2}e^{X_t}d\langle X\rangle_t \\ &= e^{X_t}f(X_t)dB_t + \frac{1}{2}e^{X_t}f(X_t)^2dt. \end{aligned}$$

So we have

$$h(x) = \frac{1}{2}e^x f(x)^2, \quad \text{and} \quad M_t - M_0 = \int_0^t e^{X_s}f(X_s)dB_s.$$

Rubrics: 4 points for the expression of h, and 3 points for stochastic integral representation of the martingale.

For each part, you will loose 1 point if you get the formulae correct but made some calculation mistakes.

(3) [7 pts]. Let $M := \max_{0 \le t \le 1} B_t$. Compute $\mathbb{E}\big[B_2 \mathbf{1}_{M \ge a}\big]$. Express your answer as a function of a.

Answer: Define the stopping time

$$T := \inf\Big\{t \ge 0 \ : \ B_t \ge a\Big\}.$$

We need to compute $\mathbb{E}[B_2 \mathbf{1}_{T \le 1}]$. By strong Markov property, given $T \le 1$ and conditionally on $(B_t : 0 \le t \le T)$, we have

$$\mathbb{E}\big[B_2 \mid (B_t : 0 \le t \le T)\big] = B_T = a.$$

Consequently, we have

$$\mathbb{E}[B_2 \mathbf{1}_{T \le 1}] = \mathbb{P}(T \le 1) \cdot \mathbb{E}\big[B_2 \mid T \le 1\big] = a\mathbb{P}(T \le 1).$$

Using reflection principle from class, we have $\mathbb{P}(T \le 1) = 2\mathbb{P}(B_1 \ge a)$. Therefore, we have

$$\mathbb{E}\big[B_2 \mathbf{1}_{M \ge a}\big] = 2a\mathbb{P}(B_1 \ge a) = \frac{2a}{\sqrt{2\pi}} \int_a^{+\infty} e^{-x^2/2} dx.$$

Rubrics: 4 points for dealing with B_2 and relating the target quantity to $\mathbb{P}(T \le 1)$.

3 points for computing $\mathbb{P}(T \le 1)$ using reflection principle.

Any other method with correct final solution gets full points.

Partial solution using other methods (e.g. using brute-force method to compute the joint distribution of B_2 and M) gets partial credit, depending on the progress.

(4) [7 pts]. Suppose that the process $(X_t)_{t\geq 0}$ satisfies $X_0 = 0$ and

$$dX_t = \frac{1}{X_t + 1} dt + dB_t$$

It is guaranteed that this process will never go below -1.

Show that $M_t = (X_t + 1)^2 - 3t$ is a martingale, and use it to compute $\mathbb{E}[\tau]$, where

$$\tau := \inf\{t > 0 : X_t \geq 2\}.$$

Answer: By Itô's formula,

$$\begin{aligned} dM_t &= 2(X_t + 1)dX_t + d\langle X\rangle_t - 3dt \\ &= 2dt + 2(X_t + 1)dB_t + dt - 3dt \\ &= 2(X_t + 1)dB_t. \end{aligned}$$

Therefore M_t is a martingale.

In order to apply OST to M_t, we need to justify the condition. Define the process

$$Y_0 = 0, \quad Y_t = \frac{1}{3}dt + dB_t.$$

Clearly, we have $X_t \geq Y_t$ before the stopping time τ. Therefore

$$\mathbb{P}(\tau \geq t) \leq \mathbb{P}\big(\forall s \in [0, t],\ Y_s \leq 2\big) \leq \mathbb{P}(Y_t \leq 2) = \Phi\Big(\frac{2 - t/3}{\sqrt{t}}\Big),$$

where Φ is the cumulative distribution function of $\mathcal{N}(0, 1)$. Consequently, $\mathbb{P}(\tau \geq t)$ is exponentially decaying as a function of t, and $|M_t| \leq 9 + 3\tau$ for $t \leq \tau$. Following the arguments in class, it is easy to show that

$$\lim_{t\to+\infty} \mathbb{E}\big[|M_t|\mathbf{1}_{\tau\geq t}\big] \leq \lim_{t\to+\infty} \mathbb{E}\big[(9 + 3\tau)\mathbf{1}_{\tau\geq t}\big] = 0.$$

Therefore, we can apply OST to obtain that

$$1 = M_0 = \mathbb{E}[M_\tau] = 9 - 3\mathbb{E}[\tau],$$

which yields $\mathbb{E}[\tau] = \frac{8}{3}$.

Rubrics: 3 points for showing martingale.

4 points for using OST to compute the expectation.

The two parts of the credit will be awarded independently (e.g. you'll get 4 points if you failed to justify martingale but succeed in computing the expectation assuming martingale).

For the OST part, you'll get 3 points if

- You get the correct answer without complete justification of OST.
- You justify the use OST correctly, but did not compute the correct answer.

Question 5. [8 pts] Let $(N(t) : t \geq 0)$ be a Poisson process with intensity parameter $\lambda > 0$. Let $0 < T_1 < T_2 < \cdots < T_{N(t)} \leq t$ be the arrival times (i.e., the marked points) within the interval $[0, t]$. Compute the quantity

$$\mathbb{E}\Big[\exp\big(-\sum_{i=1}^{N(t)} T_i\big)\Big].$$

Express it as a function of λ and t.

Answer: Given $n \in \mathbb{N}$, conditionally on $N(t) = n$, the arrival times are i.i.d. random variables from $\mathrm{Unif}([0, t])$. We have

$$\mathbb{E}\Big[\exp\big(-\sum_{i=1}^{N(t)} T_i\big) \mid N(t) = n\Big] = \mathbb{E}\big[e^{-T}\big]^n,$$

where $T \sim \mathrm{Unif}([0, t])$. And we can therefore compute

$$\mathbb{E}[e^{-T}] = \frac{1}{t}\int_0^t e^{-s} ds = \frac{1}{t}(1 - e^{-t}).$$

Substituting back,

$$\begin{aligned}
\mathbb{E}\Big[\exp\big(-\sum_{i=1}^{N(t)} T_i\big)\Big] &= \sum_{n=0}^{+\infty} \mathbb{E}\big[e^{-T}\big]^n \mathbb{P}(N(t) = n) \\
&= \sum_{n=0}^{+\infty} \Big(\frac{1 - e^{-t}}{t}\Big)^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} \\
&= e^{-\lambda t} \exp\Big(\lambda(1 - e^{-t})\Big) \\
&= \exp\big(\lambda(1 - t - e^{-t})\big).
\end{aligned}$$

Rubrics: You will lose 2 points for getting the solution idea but making calculation mistakes.

The first step (conditionally i.i.d. decomposition) is worth 4 points.

Any other method with correct solution gets full credit.

Question 6. [17 pts] Let μ be a probability distribution on the interval $[0, 1)$. Let $a_0 = 0$ and $b_0 = 1$, and recursively define

$$(a_{n+1}, b_{n+1}) := \begin{cases} (a_n, \frac{a_n+b_n}{2}), & \text{with probability } 1/2, \\ (\frac{a_n+b_n}{2}, b_n), & \text{with probability } 1/2. \end{cases}$$

In other words, at each time, we divide the current interval $[a_n, b_n)$ evenly into two parts, and randomly choose one side.

Define the process

$$X_n := 2^n \mathbb{P}_{Z\sim\mu}\big(a_n \le Z < b_n\big) \quad \text{for } n = 0, 1, 2, \cdots$$

(1) [6 pts]. Show that there exists X_∞, such that $X_n \to X_\infty$ almost surely.

Answer: Clearly, each X_n is integrable. Note that

$$\begin{aligned} &\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] \\ &= \frac{1}{2} \cdot 2^{n+1} \mathbb{P}_{Z\sim\mu}\big(a_n \le Z < \frac{a_n + b_n}{2}\big) + \frac{1}{2} \cdot 2^{n+1} \mathbb{P}_{Z\sim\mu}\big(\frac{a_n + b_n}{2} \le Z < b_n\big) \\ &= 2^n \mathbb{P}(a_n \le Z < b_n) = X_n. \end{aligned}$$

So $(X_n)_{n\ge0}$ is a martingale. Each X_n is non-negative, and consequently, there exists X_∞, such that $X_n \to X_\infty$ almost surely.

Rubrics: You'll get 2 points if you just claim martingale without verifying it.

You lose 1 point if you use martingale convergence theorem without noting the ons-side-bounded/finite expectation property.

(2) [3 pts]. Construct a probability distribution μ, such that the convergence in part (1) does not hold true in $\mathbb{L}^1$.

Answer: Let μ be an atomic measure at $\frac{1}{2}$, i.e., $\mathbb{P}_{Z\sim\mu}(Z = \frac{1}{2}) = 1$. For $n \geq 1$, we have

$$X_n = \begin{cases} 2^n & a_n = 1/2, b_n = 1/2 + 1/2^n \\ 0 & \text{otherwise.} \end{cases}$$

So we have $\mathbb{P}(X_n = 0) = 1 - 2^{-n}$. Taking the limit, we have

$$\lim_{n\to+\infty} X_n = 0 \quad \text{a.s.}$$

Each X_n satisfies $\mathbb{E}[X_n] = 1$, but the limit is 0. So $\mathbb{L}^1$ convergence does not hold.

Rubrics: Any valid example should contain some sort of atomic mass.

If you just give a correct example without justification, you get 2 points.

(3) [8 pts]. Suppose that μ has a probability density function p_μ on $[0, 1]$. Find the limiting random variable X_∞, and prove that the convergence holds true in $\mathbb{L}^1$.

[You will get half credit if you find the correct limit without proving convergence.]

Answer: By construction, the sequence of intervals $[a_n, b_n)$ will converge to a random variable $U \sim \text{Unif}([0, 1])$, and

$$X_n = \frac{1}{b_n - a_n} \int_{a_n}^{b_n} p_\mu(z)dz \to p_\mu(U),$$

which is the limiting random variable.

To show the $\mathbb{L}^1$ convergence, we note that

$$X_n = \frac{1}{b_n - a_n} \int_{a_n}^{b_n} p_\mu(z)dz = \mathbb{E}[p_\mu(U) \mid \mathcal{F}_n],$$

where $\mathcal{F}_n$ contains the information of first n-step partitions $([a_i, b_i))_{i=0}^n$.

So X_n is a Doob's martingale. By Lecture 9, we know that $(X_n)_{n\geq 0}$ is uniformly integrable, and consequently

$$X_n \xrightarrow{\mathbb{L}^1} X_\infty.$$

Rubrics: 4 points for identifying the correct limit (you only need to find the correct expression).

If you get the idea of the limit but expressed it incorrectly, you get 2 points.

For the $\mathbb{L}^1$ convergence part, partial progress gets 2 points (e.g., trying to show uniform integrability manually).