STA 447/2006 Lecture 1.

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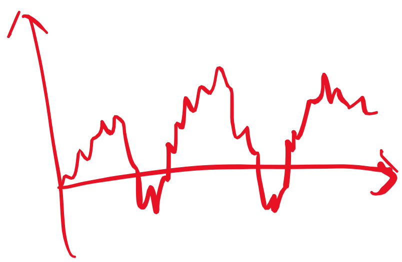
$(X_t : t \in T)$. T a period of time.

Temporal structures.

- Markov process (Discrete states).
  - $T = \mathbb{N}$ (non-neg ints). (DTMC) gambling.
  - $T = \mathbb{R}_+$ (non-neg real). (CTMC) (end of semester).
- martingales (discrete time). — gambling.
- Brownian motion and stochastic calculus.



Def. Discrete-time, discrete-state, time-homogeneous MC.

$$(X_0, X_1, X_2, \dots\dots). \qquad X_i \in S.$$

(i) State space S
(finite or countably infinite).

(ii) Initial distribution $(\nu_i)_{i\in S}$

$$X_0 \sim \nu$$

$(\mathbb{P}(X_0 = i) = \nu_i$ for $i \in S)$.

(iii). Transition probabilities.

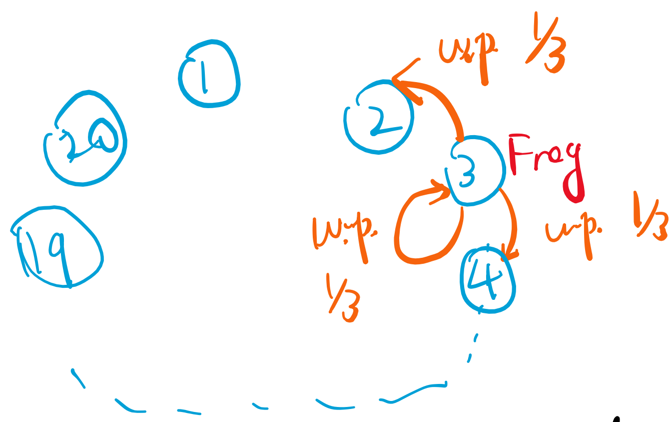
$$\boxed{P = (p_{ij})_{i,j\in S}}$$

Does not depend on time.
(In general, time-inhomogeneous MC).

$$p_{ij} = \mathbb{P}(X_{t+1} = j \mid X_t = i)$$

(for $i,j \in S$).

eg.



$X_i :=$ index of the lilypad frog sitting on.

$$S = \{1, 2, 3, \cdots, 20\}$$

$$\nu_i = \begin{cases} 1 & i = 3, \\ 0 & i \neq 3. \end{cases}$$

$$P_{ij} = \begin{cases} 1/3 & \text{when } i = j \text{ or} \\ & i = (j+1) \bmod 20 \text{ or} \\ & i = (j-1) \bmod 20 \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{bmatrix} 1/3 & 1/3 & & & 1/3 \\ 1/3 & 1/3 & \ddots & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & 1/3 \\ 1/3 & & & 1/3 & 1/3 \end{bmatrix}$$

"transition matrix".

e.g. Coin tossing.

At each time $t = 1, 2, 3, \cdots$, toss a coin.

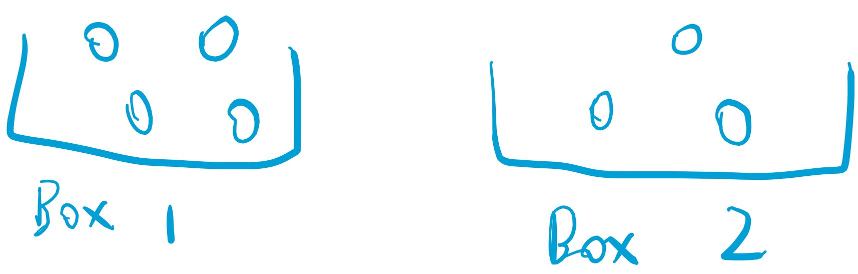
$X_t :=$ # heads in first t rounds.

$$S = \{0, 1, 2, \cdots\cdots\}$$

$\nu(0) = \nu_0 = 1$, $\quad \nu_i = 0$ (for $i \geq 1$).

$$P_{ij} = \begin{cases} 1/2 & (i = j, \text{ or } i+1 = j) \\ 0 & (\text{otherwise}). \end{cases}$$

e.g. Ehrenfest's Urn.



d balls in total.

$d = a + b$

At each time.

— Randomly select a ball (uniformly).

— Move it to the opposite side.

Start w/ a balls in 1 and b balls in 2.

$X_t :=$ # of balls in box 1 at time t

- $= \{0, 1, 2, \cdots, d\}$

- $= 1$, $\nu_i = 0$ for $(i \neq a)$

- P_{ij} $\mathbb{P}(X_{t+1} = j \mid X_t = i)$

$$X_{t+1} = \begin{cases} X_t + 1 & \text{w.p. } 1 - i/d \\ X_t - 1 & \text{w.p. } i/d \end{cases}$$

$$P_{ij} = \begin{cases} 1 - i/d & \text{when } j = i+1 \\ i/d & \text{when } j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

- Non-example if the move depends on history before time t.

Important property of MC

"Markov property".

$$\mathbb{P}(X_t = j \mid X_0 = i_0, X_1 = i_1, X_2 = i_2, \cdots X_{t-1} = i_{t-1})$$

$$= \mathbb{P}(X_t = j \mid X_{t-1} = i_{t-1}) = P_{i_{t-1}, j}$$

Corollary.

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n)$$

$$= \mathbb{P}(X_0 = i_0) \cdot \mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdot \mathbb{P}(X_2 = i_2 \mid X_0 = i_0, X_1 = i_1)$$

$$\cdots \cdots \mathbb{P}(X_n = i_n \mid X_0 = i_0, \cdots X_{n-1} = i_{n-1}).$$

$$= \nu_{i_0} \cdot P_{i_0 i_1} \cdot P_{i_1 i_2} \cdots \cdots P_{i_{n-1} i_n}.$$

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2) = \nu_{i_0} \cdot P_{i_0, i_1} \cdot P_{i_1 i_2}$$

$$\mathbb{P}(X_0 = i_0, X_2 = i_2) = \nu_{i_0} \cdot \underbrace{\sum_{i_1 \in S} P_{i_0 i_1} \cdot P_{i_1 i_2}}_{\text{Matrix multiplication.}}$$

$$\mathbb{P}(X_2 = i_2 \mid X_0 = i_0) = [P^2]_{i_0, i_2}.$$

Detour

A, B are infinite-dim matrices (countably inf S).

$$A = (a_{ij})_{i,j \in S} \qquad B = (b_{ij})_{i,j \in S}$$

$$[A \cdot B]_{ij} = \sum_{k \in S} a_{ik} b_{kj}$$

infinite sum convergence unclear in general
but convergence is true
when A, B are prob. transition matrices.

In general, for integer $k \geq 0$.

$$\mathbb{P}(X_k = j \mid X_0 = i) = [P^k]_{ij} = \mathbb{P}(X_{n+k} = j \mid X_n = i)$$

$$\mathbb{P}(X_k = j) = [\nu \cdot P^k]_j .$$

(Row vector)

Def. $p_{ij}^{(n)} := \mathbb{P}(X_n = j \mid X_0 = i)$

$$\left(= \mathbb{P}(X_{n+m} = j \mid X_m = i) \right)$$

for $i, j \in S$.

We have $P_{ij}^{(n)} = [P^n]_{ij}$.

$$P_{ij}^{(m+n)} = [P^{m+n}]_{ij}$$

$$= [P^m \cdot P^n]_{ij}$$

$$= \sum_{k \in S} P_{ik}^{(m)} \cdot P_{kj}^{(n)}$$

"Chapman-Kolmogorov Eq"

Recurrence and transience.

Def. $N(i) :=$ total number of times that MC visits state i

$$= \sum_{t=1}^{+\infty} \mathbb{1}\{X_t = i\}$$

(r.v., may be infinite in some cases).

$$f_{ij} := \mathbb{P}(N(j) \geq 1 \mid X_0 = i)$$

(Prob. reach j when starting from i).

f_{ii} : prob of returning to i if we start from i.

Def. A state i is $\begin{cases} \text{recurrent} & \text{if } f_{ii} = 1 \\ \text{transient} & \text{if } f_{ii} < 1 \end{cases}$

Fact. $\underbrace{\mathbb{P}_i(N(i) \geq k)}_{= \mathbb{P}(N(i) \geq k \mid X_0 = i)} = f_{ii}^k$

(Not to confused w/ $\mathbb{P}_X(X=x)$, not used in my lec).

"Proof." Induction. $k=1$. By def.

Assuming the claim is true for $k-1$.

$$\mathbb{P}_i(N(i) \geq k) = \boxed{\mathbb{P}_i(N(i) \geq k \mid N(i) \geq k-1)} \cdot \underbrace{\mathbb{P}_i(N(i) \geq k-1)}_{= f_{ii}^{k-1} \text{ by induction.}}$$

Need to show $= f_{ii}$.

$$\begin{array}{cccccccc} X_0 & X_1, & \dots, & X_{\tau^{(1)}}, & X_{\tau+1} & \dots & X_{\tau^{(2)}} & \dots \\ \| & & \dots & \| & & \dots & \| & \dots \\ i & & & i & & & i & \end{array}$$

Past | Future

$$\begin{array}{cc} \dots & X_{\tau^{(k-1)}} \\ & \| \\ & i \end{array}$$

$$\begin{aligned} &\mathbb{P}\left(\text{visit } i \text{ again} \mid X_0\, X_1, \dots\, X_{\tau^{(k-1)}}\right) \\ &= \mathbb{P}\left(\text{visit } i \text{ again} \mid X_{\tau^{(k-1)}} = i\right) \\ &= \mathbb{P}\left(\text{visit } i \text{ again} \mid X_0 = i\right) = f_{ii}. \end{aligned}$$

"Strong Markov property".

Let τ be the hitting time of state i, then we have

$$(X_\tau, X_{\tau+1}, \dots) \overset{d}{=} (\underset{\overset{\|}{i}}{X_0}, X_1, \dots).$$

Also applies to

$$\tau^{(k)} = k\text{-th hitting time of } i.$$

By strong Markov property,

$$\mathbb{P}_i\left(N(i) \geq k \mid N(i) \geq k-1 \right) = f_{ii}.$$

$$\boxed{\begin{array}{l}\textbf{Def.}\ \text{hitting time of } i \\ \tau_i := \inf\{t \geq 1 : X_t = i\}\end{array}}$$

Corollary.

$$\mathbb{P}_i\left(N(j) \geq k \right) = f_{ij} \cdot f_{jj}^{k-1}.$$

$$\underset{\displaystyle i}{\overset{\displaystyle X_0}{\|}} \cdots \left[\underset{\displaystyle j}{\overset{\displaystyle X_{\tau_j}}{\|}} \cdots\cdots\cdots \right]$$

Fresh new MC starting from j.

$$\mathbb{P}_i(N(j) \geq k)$$

$$= \mathbb{P}_i(N(j) \geq 1) \cdot \mathbb{P}_i(N(j) \geq k \mid N(j) \geq 1)$$

$$= f_{ij} \cdot \mathbb{P}_j(N(j) \geq k-1)$$

$$= f_{ij} \cdot f_{jj}^{k-1}$$

Notation

$$\mathbb{P}_i(N(j) \geq k \mid N(j) \geq 1)$$

$$= \mathbb{P}(N(j) \geq k \mid N(j) \geq 1, X_0 = i).$$

$$= \mathbb{P}(N(j) \geq k \mid N(j) \geq 1)$$

$$= \mathbb{P}(N(j) \geq k-1 \mid X_0 = j).$$

Conclusion. (starting from i)

If $f_{ii}=1$ then $N(i)=+\infty$ w.p. 1

If $f_{ii}<1$ then

$$\mathbb{P}(N(i)=k) = f_{ii}^k - f_{ii}^{k+1}$$

(Geometric distribution).

e.g.

$$\mathbb{E}_i[N(j)] = \begin{cases} \frac{f_{ij}}{1-f_{jj}} & (f_{jj}<1) \\ +\infty & (f_{jj}=1,\ f_{ij}>0) \\ 0 & (f_{ij}=0). \end{cases}$$

Question: can we determine recurrence/transience by computing sth. using $(p_{ij})_{ij\in S}$.

Recurrence State Theorem.

State i is recurrent if and only if

$$\sum_{n=1}^{+\infty} P_{ii}^{(n)} = +\infty.$$

Proof.

$$\sum_{n=1}^{+\infty} P_{ii}^{(n)}$$

$$= \sum_{n=1}^{+\infty} \mathbb{P}_i(X_n = i)$$

$$= \sum_{n=1}^{+\infty} \mathbb{E}_i\left[\mathbb{1}_{X_n=i}\right].$$

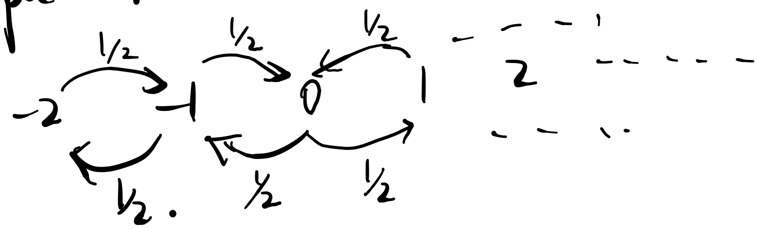
(Fubini - Tonelli)

$$= \mathbb{E}_i\left[\sum_{n=1}^{+\infty} \mathbb{1}_{X_n=i}\right].$$

$$= \mathbb{E}_i[N(i)] = \begin{cases} \dfrac{f_{ii}}{1-f_{ii}}. & f_{ii} < 1 \\ +\infty & f_{ii} = 1. \end{cases}$$

(By-product, able to compute f_{ii} from $(P_{ij})_{i,j\in S}$).

e.g. Simple Random Walk.

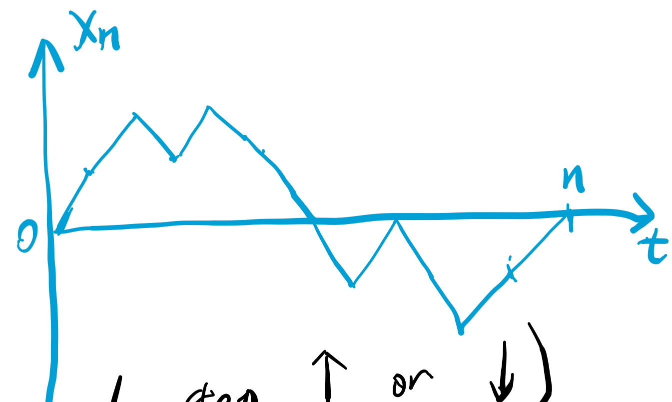


$$p_{i(i+1)} = p_{i(i-1)} = \frac{1}{2} \quad \text{for } i \in S = \mathbb{Z}.$$

Question: whether $f_{00} = 1$?

Suffices to check (by recurrent state thm)

$$\sum_{n=1}^{+\infty} p_{00}^{(n)}.$$



$p_{00}^{(n)} = 0$ (if n is odd).

Total #paths $= 2^n$ (for each step, ↑ or ↓).

paths that go back to 0 at time n $= \binom{n}{n/2} = \frac{n!}{(n/2)!^2}$.

(# paths $\frac{n}{2}$ ↑'s and $\frac{n}{2}$ ↓'s for n steps)

Stirling's approx $n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$.

$$p_{00}^{(n)} \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}. \qquad \sum_{n=1}^{+\infty} p_{00}^{(n)} = +\infty$$

So 0 is recurrent.