STA447/2006 Lecture 10

Goal: make sense of "calculus" for BM.

$$\int_0^t Y_s \, dB_s$$

e.g. gambling strategy.

Recall in discrete time (MG convergence proof)

Fact: $(X_t)_{t=0,1,2,\cdots}$ is discrete-time MG

If $(Y_t)_{t=0,1,2,\cdots}$ is a gambling strategy.

Each Y_t is determined by $X_1, X_2, \cdots, X_{t-1}$, unif bdd

then $M_t = \sum_{k=1}^{t} Y_k (X_k - X_{k-1})$ is also a martingale

M: stochastic integration of $(Y_t)_{t=0,1,2,\cdots}$ w.r.t $(X_t)_{t=0,1,2,\cdots}$

How about the second moment?

Assume that $(X_t)_{t\geq 0}$ is SSRW.

$$\varepsilon_t = X_t - X_{t-1} \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$$

$$\mathbb{E}[M_t^2] = \mathbb{E}\left[\left(\sum_{k=1}^{t} Y_k \varepsilon_k\right)^2\right]$$ (also extends to independent normal).

$$= \sum_{k,l=1}^{t} \mathbb{E}[Y_k Y_l \varepsilon_k \varepsilon_l].$$

- For $k=l$, $\mathbb{E}[Y_k Y_l \varepsilon_k \varepsilon_l] = \mathbb{E}[Y_k^2 \cdot \varepsilon_k^2] = \mathbb{E}[Y_k^2]$.

  $\longrightarrow$ conditionally zero-mean.

- For $k<l$, $\mathbb{E}[\underbrace{Y_k Y_l \varepsilon_k}_{\text{Known at time } t-1} \varepsilon_l]$

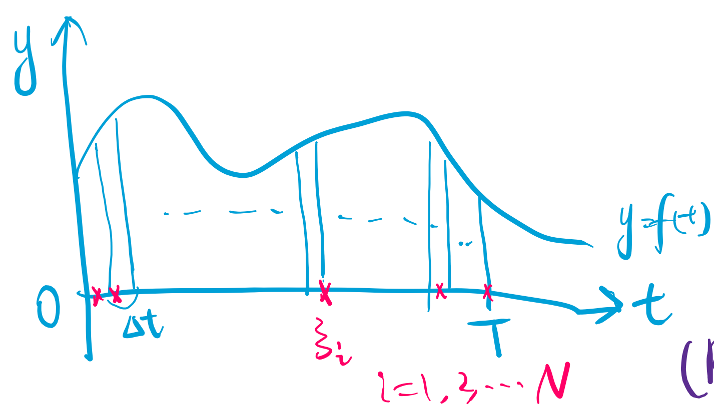
$$= \mathbb{E}\left[Y_k Y_l \varepsilon_k \, \mathbb{E}[\varepsilon_l \mid \mathcal{F}_{l-1}]\right] = 0.$$

- Similarly, for $k>l$, $\cdots\cdots = 0$ (symmetry).

So. $\mathbb{E}[M_t^2] = \sum_{k=1}^{t} \mathbb{E}[Y_k^2]$.

How about BM?

Recall: Riemann integral.



$$\int_0^T f(t)\,dt = \lim_{\Delta t \to 0} \sum_{i=1}^{N} f(\xi_i)\,\Delta t$$

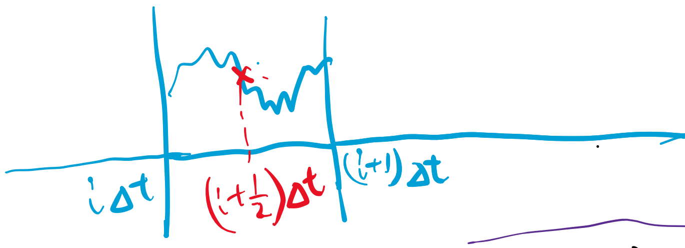
(Regardless of the choice of ξ_i's).

Idea: mimic this argument

$$\int_0^T Y_s \, dB_s \overset{?}{=} \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} Y_{\xi_i}\left(B_{(i+1)\Delta t} - B_{i\Delta t}\right).$$

("Stieltjes integral").

e.g. Let $Y_t = B_t$, $\int_0^t B_s \, dB_s$?



$N = T/\Delta t$

$$\lim_{\Delta t \to 0} \sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{(i+\frac{1}{2})\Delta t}$$

$$= \sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) B_{i\Delta t}$$

Discrete time MG (from gambling strategy).

$$+ \underbrace{\sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot \left(B_{(i+\frac{1}{2})\Delta t} - B_{i\Delta t}\right)}$$

Residual ($\to 0$ in Riemann integral)

Residual:

each term $= \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot \left(B_{(i+\frac{1}{2})\Delta t} - B_{i\Delta t}\right)$

independent, identically distributed.

$$\mathbb{E}[\text{term}_i] = \mathbb{E}\left[\left(B_{(i+\frac{1}{2})\Delta t} - B_{i\Delta t}\right)^2\right] = \frac{\Delta t}{2}.$$

$$\#\text{ terms} = \frac{T}{\Delta t}.$$

By SLLN, residual $\longrightarrow \frac{T}{2}$ w.p. 1

Conclusion. by taking mid-points,

$$\lim_{\Delta t \to 0} \sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) B_{(i+\frac{1}{2})\Delta t} \to MG + \frac{T}{2}.$$

(Remark: this is called "Stratonovich integral", preserves chain rule, etc, but not martingales).

e.g. We know $\int_0^t f(s)\, df(s) = \frac{1}{2}f(t)^2 - \frac{1}{2}f(0)^2$ for cts. diff. function f.

Indeed, for BM, Stratonovich integral $\int_0^T B_t \circ dB_t = \frac{1}{2}B_T^2$

We'll show later. $\leftarrow (=) \int_0^T B_t\, dB_t + \frac{1}{2}T.$

(As we've seen, $\left(\frac{1}{2}B_t^2 - \frac{1}{2}t\right)_{t\geq 0}$ is a MG).

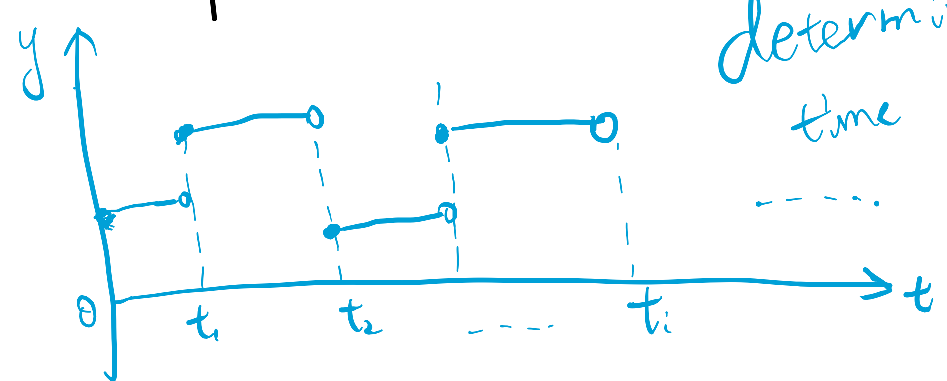
Roadmap

— Define stoch integral for piecewise constant proc. (gambling strategy for DTMGs)

— Show some convergence

Piecewise constant processes. (need to be adapted).

$t_1, t_2, \cdots$ are deterministic time points.



$$Y_s = Y^{(i)} \quad \text{for} \quad s \in [t_i, t_{i+1}).$$

$$\int_0^T Y_t \, dB_t := \sum_{i=0}^{N-1} Y^{(i)} (B_{t_{i+1}} - B_{t_i}).$$

(1) MG; and satisfies 2^{nd} moment identity

(2) $$\mathbb{E}\left[\left|\int_0^T Y_t \, dB_t\right|^2\right] = \sum_{i=0}^{N-1} \mathbb{E}\left[|Y^{(i)}|^2\right] \cdot (t_{i+1} - t_i)$$

$$= \int_0^T \mathbb{E}\left[|Y_t|^2\right] dt.$$

"Itô's isometry".

(3) For deterministic $a, b \in \mathbb{R}$,

$$\int_0^T \left(a Y_t^{(1)} + b Y_t^{(2)}\right) dB_t = a \int_0^T Y_t^{(1)} dB_t + b \int_0^T Y_t^{(2)} dB_t$$

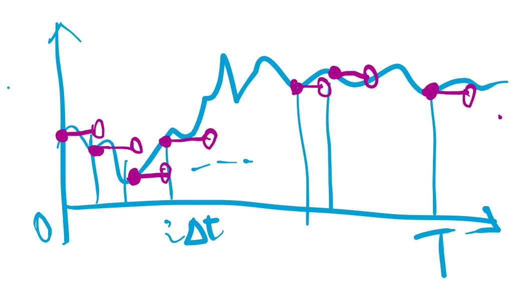
Now we define Itô's integral for general processes.

Given $(Y_t)_{t \geq 0}$ "adapted"

Y_t depends only on $(B_s)_{0 \leq s \leq t}$.

$$Y_t^{(n)} := Y_{i\Delta t} \quad \text{for} \quad t \in [i\Delta t, (i+1)\Delta t)$$

where $\Delta t = T/n$.



Fact (we'll not give a proof):

If $(Y_t)_{t\geq 0}$ is adapted, and continuous

(can be extended to right-cts-left-limit processes)

$$\lim_{n\to+\infty} \int_0^T \mathbb{E}\left[|Y_t - Y_t^{(n)}|^2\right] dt = 0.$$

As a result

$$\lim_{m,n\to+\infty} \int_0^T \mathbb{E}\left[|Y_t^{(m)} - Y_t^{(n)}|^2\right] dt = 0.$$

and by Itô's isometry for piecewise const,

$$\mathbb{E}\left[\left|\int_0^T Y_t^{(m)} dB_t - \int_0^T Y_t^{(n)} dB_t\right|^2\right] \longrightarrow 0$$

as $m, n \to +\infty$.

So $\left(\int_0^T Y_t^{(n)} dB_t\right)_{n\geq 0}$ forms a Cauchy seq

in $\mathbb{L}^2$, and the limit exists.

We define Itô's integral $\int_0^T Y_t dB_t$

as the $\mathbb{L}^2$ limit of this sequence.

Easy to verify: $\int_0^T Y_t dB_t$ inherits all the

nice properties.

$$\begin{cases} \left(\int_0^t Y_s \, dB_s\right)_{t\geq 0} \text{ is a martingale} \\ \mathbb{E}\left[\left(\int_0^t Y_s \, dB_s\right)^2\right] = \int_0^t \mathbb{E}[Y_s^2] \, ds. \\ \int_0^t \left(a Y_s^{(1)} + b Y_s^{(2)}\right) dB_s = a \cdot \int_0^t Y_s^{(1)} dB_s + b \cdot \int_0^t Y_s^{(2)} dB_s. \end{cases}$$

Recall. Fundamental thm of calculus

- $$\int_a^b f'(x)\,dx = f(b) - f(a).$$
- (in differential form)

$$df(x) = f'(x)\,dx.$$

Question:

$$df(B_t) \overset{?}{=} f'(B_t)\,dB_t$$

Answer: no. (for Itô's integral)

e.g. $f(x) = \frac{1}{2}x^2$

$$\int_0^T f'(B_t)\,dB_t \neq \frac{1}{2}B_t^2$$

Can we have a systematic way to compute this?

Thm. If f is a twice continuously differentiable,

$$f(B_t) - f(B_0) = \underbrace{\int_0^t f'(B_s)\,dB_s}_{\text{Itô}} + \underbrace{\frac{1}{2}\int_0^t f''(B_s)\,ds}_{\text{Riemann}}$$

(In differential form)

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2} f''(B_t)\,dt.$$

Examples.

1. $f(x) = \frac{1}{2}x^2$ $\quad df(B_t) = B_t\,dB_t + \frac{1}{2}dt$

So $\int_0^t B_s\,dB_s = \frac{1}{2}B_t^2 - \frac{1}{2}t.$

(Recall, gambler's ruin — verify $\left(\frac{B_t^2}{2} - \frac{1}{2}t\right)_{t\geq 0}$ MG manually)

Now we can verify MG's following straightforward derivation.

2. $f(x) = e^x$.

$$df(B_t) = e^{B_t}\,dB_t + \frac{1}{2}e^{B_t}\,dt$$

So $e^{B_t} - \frac{1}{2}\int_0^t e^{B_s}\,ds - 1 = \int_0^t e^{B_s}\,dB_s$

is a martingale.

Deriving Itô's formula.

Recall, fundamental thm. calc.

$$f(b) - f(a) = \sum_{i=0}^{N-1} \left(f(t_{i+1}) - f(t_i)\right)$$

where $a = t_0 < t_1 < \cdots < t_{N-1} = b$. equi-spaced

$$f(t_{i+1}) - f(t_i) = f'(t_i) \cdot (t_{i+1} - t_i) + o(\Delta t).$$

So we get

$$f(b) - f(a) = \sum_{i=0}^{N-1} f'(t_i)(t_{i+1} - t_i) + o(N \cdot \Delta t)$$

$(N \cdot \Delta t = b - a)$ So $o(N \cdot \Delta t) \to 0$.

For Itô:

$$f(B_t) - f(B_0) = \sum_{i=0}^{N-1} f(B_{t_{i+1}}) - f(B_{t_i})$$

$$f(B_{t_{i+1}}) - f(B_{t_i})$$

$$= f'(B_{t_i}) \cdot (B_{t_{i+1}} - B_{t_i}) + \frac{1}{2} f''(B_{t_i}) \cdot (B_{t_{i+1}} - B_{t_i})^2$$

$$+ o\left((B_{t_{i+1}} - B_{t_i})^2\right) = o_p(\Delta t)$$

So we get

$$f(B_T) - f(B_0)$$

$$= \underbrace{\sum_{i=0}^{N-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i})}_{\to \int_0^T f'(B_t)\,dB_t} + \frac{1}{2} \sum_{i=0}^{N-1} f''(B_{t_i})(B_{t_{i+1}} - B_{t_i})^2$$

$$+ o(N \cdot \Delta t) \to 0.$$

How about the correction term?

e.g. If f is quadratic $f(x)=\frac{1}{2}x^2$

$f''=1$.

$$\frac{1}{2}\sum_{i=0}^{N-1}\left(B_{t_{i+1}}-B_{t_i}\right)^2 \xrightarrow{a.s.} \frac{T}{2}.$$

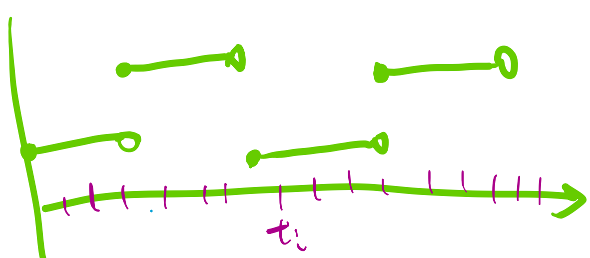
(we have seen this already).

Roadmap: constant $\xrightarrow[(i)]{}$ piecewise const $\xrightarrow[(ii)]{}$ cts function.

Step (i): Consider

$$\lim_{N\to+\infty}\sum_{i=0}^{N-1} g(t_i)\left(B_{t_{i+1}}-B_{t_i}\right)^2 = \int_0^T g(t)\,dt = \sum_{\substack{k\geq 0\\ \tau_{k+1}\leq T}} g(\tau_k)\cdot(\tau_{k+1}-\tau_k)$$

where g depends on $(B_t)_{0\leq t\leq t_i}$



$g(t)=g(\tau_k)$ for $t\in[\tau_k,\tau_{k+1})$.

Proof: Apply const func. arguments to $[\tau_k,\tau_{k+1})$, conditionally on $(B_t)_{0\leq t\leq \tau_k}$.

Step (ii): given $f \in C^2$.

We let $\tau_k = \frac{k}{n} T$ for some n

and let $g^{(n)}(t) = \frac{1}{2} f''(B_{\tau_k})$ for $t \in [\tau_k, \tau_{k+1})$.

$f'' \in C$, $(B_t)_{t \geq 0}$ cts

$$\Rightarrow \sup_{0 \leq t \leq T} \left| g(t) - \frac{1}{2} f''(B_t) \right| \to 0$$

(as $n \to +\infty$).

So we get

$$\left| \lim_{N \to +\infty} \sum_{i=0}^{N-1} \frac{1}{2} f''(B_{t_i}) \left(B_{t_{i+1}} - B_{t_i} \right)^2 - \underbrace{\lim_{N \to +\infty} \sum_{i=0}^{N-1} g^{(n)}(t) \cdot \left(B_{t_{i+1}} - B_{t_i} \right)^2}_{= \int_0^T g^{(n)}(t)\, dt} \right| \to 0$$

So the limit $= \lim_{n \to +\infty} \int_0^T g^{(n)}(t)\, dt = \int_0^T \frac{1}{2} f''(B_t)\, dt$.

In general, we define

$$[X]_t := \lim_{N \to +\infty} \sum_{i=0}^{N-1} \left(X_{t_{i+1}} - X_{t_i} \right)^2$$

where $t_i = \frac{i t}{N}$

for $i = 0, 1, 2, \ldots N$.

"Quadratic variation" of $(X_t)_{t\geq 0}$.

e.g. $(B_t)_{t\geq 0}$, $[B]_t = t$.