Recap:

— Itô's integration

Define $\int_0^T X_t \, dB_t$ as L^2 limit of

$$\sum_{i=0}^{N-1} X_{i\Delta t} \cdot \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right).$$

a martingale.

— Itô's formula.

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2} f''(B_t)\,dt$$ "differential form"

$$f(B_T) = f(B_0) + \int_0^T f'(B_t)\,dB_t + \frac{1}{2}\int_0^T f''(B_t)\,dt$$ "integration form"!

(Remark: differential form is just a shorthand notation for the integration form).

Next step: generalizations

- $f(t, B_t)$, $f(t, Z_t)$
- $dZ_t = X_t\,dt + Y_t\,dB_t$, study $f(Z_t)$
- Multivariate extension.

Suppose a continuous-time process $(Z_t)_{t\geq 0}$ satisfies $dZ_t = X_t\,dt + Y_t\,dB_t$

(for any $T>0$, $Z_T - Z_0 = \int_0^T X_t\,dt + \int_0^T Y_t\,dB_t$)

This covers a lot of important processes.

e.g. $Z_t = f(B_t)$, then by Itô's formula,

$$dZ_t = f'(B_t)\,dB_t + \frac{1}{2}f''(B_t)\,dt$$

e.g. Stochastic differential equations.

$$dZ_t = -Z_t\,dt + dB_t$$

(in general) $dZ_t = -\nabla U(Z_t)\,dt + dB_t$.

As before, we Taylor-expand f. $\left(N = \frac{T}{\Delta t}\right)$.

$$f(Z_T) - f(Z_0) = \sum_{i=0}^{N-1}\left(f(Z_{(i+1)\Delta t}) - f(Z_{i\Delta t})\right)$$

$$= \sum_{i=0}^{N-1}\Bigg[\underbrace{f'(Z_{i\Delta t})\cdot\left(Z_{(i+1)\Delta t} - Z_{i\Delta t}\right)}_{①} + \underbrace{\frac{1}{2}f''(Z_{i\Delta t})\cdot\left(Z_{(i+1)\Delta t} - Z_{i\Delta t}\right)^2}_{②} + \boxed{o\left(\left|Z_{(i+1)\Delta t} - Z_{i\Delta t}\right|^2\right)}\Bigg]$$

Since $Z_{(i+1)\Delta t} - Z_{i\Delta t} = \int_0^{\Delta t} X_{i\Delta t+s}\,ds + \int_0^{\Delta t} Y_{i\Delta t+s}\,dB_s$

$$\mathbb{E}\left[\left|Z_{(i+1)\Delta t} - Z_{i\Delta t}\right|^2\right] \leq 2\,\mathbb{E}\left[\left(\int_0^{\Delta t} X_{i\Delta t+s}\, ds\right)^2\right] + 2\int_0^{\Delta t} \mathbb{E}\left[Y_{i\Delta t+s}^2\right] ds$$

So, $o\left(\left|Z_{(i+1)\Delta t} - Z_{i\Delta t}\right|^2\right) = o(\Delta t)$.

$$\sum_{i=0}^{N-1} o(\Delta t) = o(N\cdot\Delta t) = o(1).$$

So the residual $\to 0$.

Term ①: Riemann integral + Itô's integral.

$$\lim_{N\to+\infty} \sum_{i=0}^{N-1} f'(Z_{i\Delta t})\cdot\left\{\underbrace{\int_{i\Delta t}^{(i+1)\Delta t} X_t\, dt}_{\downarrow\ \int_0^T f'(Z_t)\cdot X_t\, dt} + \underbrace{\int_{i\Delta t}^{(i+1)\Delta t} Y_t\, dB_t}_{\downarrow\ \int_0^T f'(Z_t)\, Y_t\, dB_t}\right\}$$

So Term 1 $\to \int_0^T f'(Z_t)\cdot(X_t\, dt + Y_t\, dB_t) = \int_0^T f'(Z_t)\, dZ_t$.

Term 2:

$$\lim_{N\to+\infty} \sum_{i=0}^{N-1} f''(Z_{i\Delta t})\cdot\left(Z_{(i+1)\Delta t} - Z_{i\Delta t}\right)^2$$

$$= \int_0^T f''(Z_t)\, Y_t^2\, dt.$$

As before, define quadratic variation process.

$$\langle Z \rangle_t := \lim_{n \to +\infty} \sum_{i=0}^{n-1} \left(Z_{\frac{(i+1)t}{n}} - Z_{\frac{it}{n}} \right)^2 .$$

(Note: notation change from last time, we'll use $\langle \cdot \rangle_t$ throughout the rest of course).

(increasing, and cts in t).

Using the piecewise const approximation, one can show

$$\lim_{n \to +\infty} \sum_{i=0}^{n-1} f''(Z_{i\Delta t}) \left(Z_{(i+1)\Delta t} - Z_{i\Delta t} \right)^2 = \int_0^T f''(Z_t) \, d\langle Z \rangle_t$$

Need to show $d\langle Z \rangle_t = Y_t^2 \, dt$.

Indeed:

$$Z_{(i+1)\Delta t} - Z_{i\Delta t} = \underbrace{\int_0^{\Delta t} X_{i\Delta t + s} \, ds}_{O(\Delta t)} + \underbrace{\int_0^{\Delta t} Y_{i\Delta t + s} \, dB_s}_{O(\sqrt{\Delta t}).}$$

$$\left(Z_{(i+1)\Delta t} - Z_{i\Delta t} \right)^2 = \left(\int_0^{\Delta t} Y_{i\Delta t + s} \, dB_s \right)^2 + O(\Delta t^2) + O(\Delta t^{3/2})$$

$$\sum_{i=0}^{N-1} \left(Z_{(i+1)\Delta t} - Z_{i\Delta t} \right)^2 = \underbrace{\sum_{i=0}^{N-1} \left(\int_0^{\Delta t} Y_{i\Delta t + s} \, dB_s \right)^2}_{\downarrow \ \int_0^t Y_s^2 \, ds.} + \underbrace{O(N \cdot \Delta t^{3/2})}_{\to 0}$$

since $\Delta t = \frac{T}{N}$

Conclusion:

$$df(Z_t) = f'(Z_t)\,dZ_t + \frac{1}{2}f''(Z_t)\,d\langle Z\rangle_t$$

$$= f'(Z_t)X_t\,dt + f'(Z_t)Y_t\,dB_t + \frac{1}{2}f''(Z_t)Y_t^2\,dt.$$

e.g. $f(x) = x^2$

$$d(Z_t^2) = 2Z_t\,dZ_t + d\langle Z\rangle_t$$

$$= 2Z_t(X_t\,dt + Y_t\,dB_t) + Y_t^2\,dt.$$

Suppose that $dZ_t^{(i)} = X_t^{(i)}\,dt + Y_t^{(i)}\,dB_t$ for $i \in \{1,2\}$.

We can apply formula for $d(Z_t^2)$ to $\frac{Z_t^{(1)} + Z_t^{(2)}}{2}$ and $\frac{Z_t^{(1)} - Z_t^{(2)}}{2}$

$$d\left(Z_t^{(1)} \cdot Z_t^{(2)}\right) = d\left(\left(\frac{Z_t^{(1)} + Z_t^{(2)}}{2}\right)^2\right) - d\left(\left(\frac{Z_t^{(1)} - Z_t^{(2)}}{2}\right)^2\right)$$

$$= Z_t^{(1)}\,dZ_t^{(2)} + Z_t^{(2)}\,dZ_t^{(1)} + Y_t^{(1)}Y_t^{(2)}\,dt.$$

"polarization trick".

$$\int_0^t Y_s^{(1)}Y_s^{(2)}\,ds = \lim_{n\to+\infty}\sum_{i=0}^{n-1}\left(Z^{(1)}_{\frac{(i+1)t}{n}} - Z^{(1)}_{\frac{it}{n}}\right)\cdot\left(Z^{(2)}_{\frac{(i+1)t}{n}} - Z^{(2)}_{\frac{it}{n}}\right).$$

$$\langle Z^{(1)}, Z^{(2)} \rangle_t := \lim \left[\cdots \right]$$

Following this notation, we have product rule.

$$d\left(Z_t^{(1)} \cdot Z_t^{(2)}\right) = Z_t^{(2)} dZ_t^{(1)} + Z_t^{(1)} dZ_t^{(2)} + d\langle Z^{(1)}, Z^{(2)} \rangle_t.$$

How about $f(t, Z_t)$?

$$f(t, Z_t) - f(0, Z_0) = \sum_{i=0}^{N-1} f\left((i+1)\Delta t, Z_{(i+1)\Delta t}\right) - f\left(i\Delta t, Z_{i\Delta t}\right)$$

$$f\left(i\Delta t, Z_{(i+1)\Delta t}\right) - f\left(i\Delta t, Z_{i\Delta t}\right)$$

similar to time-homogeneous case

$f(t,x)$

$$\boxed{f\left((i+1)\Delta t, Z_{(i+1)\Delta t}\right) - f\left(i\Delta t, Z_{(i+1)\Delta t}\right)}$$

just time derivative.

Conclusion.

$$df(t, Z_t) = \frac{\partial f}{\partial t}(t, Z_t)\, dt + \frac{\partial f}{\partial x}(t, Z_t)\, dZ_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(t, Z_t) \cdot d\langle Z \rangle_t.$$

e.g. $f(t, x) = \exp(at + bx)$.

$$Z_t = e^{at + bB_t}.$$

$$dZ_t = a\cdot e^{at+bB_t}dt + b\cdot e^{at+bB_t}dB_t + \frac{b^2}{2}e^{at+dB_t}dt.$$

$$= \left(a+\frac{b^2}{2}\right)\cdot Z_t\,dt + b\cdot Z_t\,dB_t.$$

- Special case: when $a = -b^2/2$.

$Z_t = \exp\left(-\frac{b^2 t}{2} + bB_t\right)$ is a martingale.

- SDE $\quad dZ_t = rZ_t dt + bZ_t\,dB_t$

Let $a = r - \frac{b^2}{2}$, and we get

the solution $Z_t = \exp\left(bB_t + \left(r-\frac{b^2}{2}\right)t\right)$.

eg. (OU process). $\quad f(t,x) = e^t\cdot x$

$$dZ_t = -Z_t\,dt + dB_t,\quad Z_0 = 0.$$

$\left(\frac{\partial^2 f}{\partial x^2} = 0\right)$

$$d(e^t Z_t) = e^t dZ_t + e^t Z_t\,dt + 0$$

$$= e^t(-Z_t\,dt + dB_t) + e^t Z_t\,dt$$

$$= e^t\,dB_t.$$

So $\quad Z_t = e^{-t}\int_0^t e^s dB_s$

$$\sim \mathcal{N}\left(0,\ e^{-2t}\int_0^t e^{2s}\,ds\right).$$

Multivariate extension.

$$B_t = \left(B_t^{(1)}, B_t^{(2)}, \dots, B_t^{(d)}\right)$$

where $B_t^{(1)}, B_t^{(2)}, \dots B_t^{(d)}$ are iid BM's.

Itô's formula.

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)\,dt + \nabla_x f(t, B_t)^T dB_t + \frac{1}{2}\Delta_x f(t, B_t)\,dt$$

where $\Delta_x f(t,x) = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}(t,x)$.

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e.g. $\quad X_0 \sim \mu$ (on $\mathbb{R}^d$)

$$X_t = X_0 + B_t.$$

Let $p_t(x)$ be the density of X_t at point x.

For $s < t$,

$$p_t(x) = \int p_s(y)\cdot \phi\left(\frac{y-x}{t-s}\right)dy$$

(ϕ is d-dim Gaussian density).

$$= \mathbb{E}\left[p_s(x + B_t - B_s)\right].$$

$$\frac{\partial p_t}{\partial t}(x) = \frac{\partial}{\partial t}\mathbb{E}\left[p_s(x+B_t-B_s)\right]$$

By Itô's formula,

$$d p_s(x+B_t-B_s) = \nabla_x p_s(x+B_t-B_s)^T dB_t + \frac{1}{2}\Delta_x p_s(x+B_t-B_s)\, dt$$

$$p_t(x) - p_s(x)$$

$$= \mathbb{E}\left[p_s(x+B_t-B_s)\right] - p_s(x)$$

$$= \mathbb{E}\left[\int_s^t \nabla_x p_s(x+B_r-B_s)^T dB_r + \frac{1}{2}\Delta_x p_s(x+B_r-B_s)\, dr\right]$$

$$= \frac{1}{2}\mathbb{E}\left[\int_s^t \Delta_x p_s(x+B_r-B_s)\, dr\right]$$

$t \to s^+$. $$\frac{\partial p_t}{\partial t}(x) = \frac{1}{2}\Delta_x p_t(x).$$

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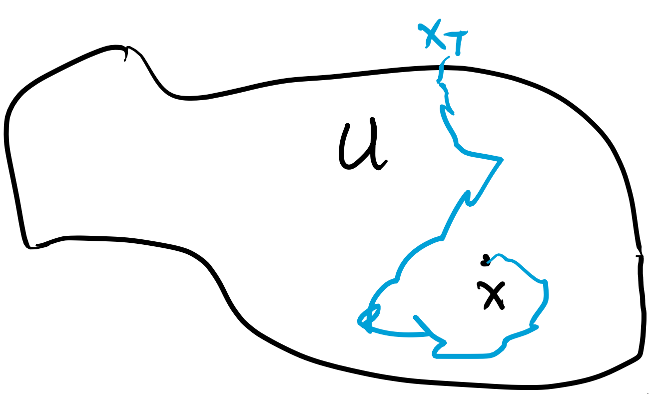
eg.



$U \subseteq \mathbb{R}^d$ a bounded region w/ smooth boundary.

In PDE literature, Laplace's equation

$$\begin{cases} \Delta f(x) = 0 & \text{for } x \in U \\ f = g & \text{at boundary } \partial U \end{cases}$$

Suppose we have a solution.

Starting from some $x \in \text{interior}(U)$.

$$X_t = x + B_t$$

$$f(X_t) = f(x) + \int_0^t \nabla f(X_s)^T dB_s \cancel{+ \frac{1}{2}\int_0^t \Delta f(X_s) ds}.$$

Let $T :=$ hitting time of boundary

(stopping time, $\mathbb{P}(T < +\infty) = 1$).

Before T, $\Delta f(X_s) = 0$ (by PDE).

$$f(X_t) = f(x) + \int_0^t \nabla f(X_s)^T dB_s,$$ is an MG

Under bdd/smooth assumptions, we can apply OST.
(e.g. when f is bdd in U)

$$\mathbb{E}[f(X_T)] = f(x).$$

This gives us explicit solution to the PDE

$$f(x) = \mathbb{E}[g(x + B_T)].$$

(c.f. gambler's ruin problem in 1-D).

Application of the Lap Eq. $(d \geq 2)$.

$$U := \{x \in \mathbb{R}^d : R_1 \leq \|x\|_2 \leq R_2\}.$$

$$g \equiv 1 \quad \text{on } \{x : \|x\|_2 = R_2\}$$

$$g \equiv 0 \quad \text{on } \{x : \|x\|_2 = R_1\}$$

$$f(x) = \mathbb{P}_x(\text{BM hits outer sphere first}).$$

The solution (know to be unique)

$$f(x) = \begin{cases} \dfrac{\log \|x\|_2 - \log R_1}{\log R_2 - \log R_1} & (d=2) \\[2ex] \dfrac{R_1^{2-d} - \|x\|_2^{2-d}}{R_1^{2-d} - R_2^{2-d}} & (d \geq 3). \end{cases}$$

By taking $R_2 \to +\infty$ while R_1, x fixed.

$$\lim_{R_2 \to +\infty} f(x) = \begin{cases} 0 & (d=2) \\ 1 - \left(\dfrac{\|x\|_2}{R_1}\right)^{2-d} & (d \geq 3) \end{cases}$$

So we got, for $R_1 < \|x\|_2$.

$$\mathbb{P}_x\left(\text{never hit } \mathbb{B}(0, R_1)\right) = \begin{cases} 0 & (d=2) \\ 1-\left(\frac{R_1}{\|x\|_2}\right)^{d-2} & (d\geq 3). \end{cases}$$

(c.f. recurrence of SRW in $d=2$/ $d\geq 3$).

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eg. Feynman-Kac formula.

Suppose that Z satisfies SDE

$$dZ_t = a(Z_t)dt + b(Z_t)dB_t.$$

"diffusion process"

Consider a linear PDE.

$$\frac{\partial V}{\partial t}(t,x) = \frac{1}{2}b^2(x)\cdot\frac{\partial^2 V}{\partial x^2}(t,x) + a(x)\cdot\frac{\partial V}{\partial x}(t,x) + u(x)\cdot V(t,x).$$

$$V(0,x) = f(x)$$

$$V(t,x) = \mathbb{E}_x\left[f(Z_t) \cdot \exp\left(\int_0^t u(Z_s)\,ds \right)\right].$$

is the solution.

Idea: apply Itô's formula to

$$M_s = V(t-s, Z_s) \cdot \exp\left(\int_0^s u(Z_r)\,dr \right).$$