STA 447/2006 Lecture 12

Final exam logistics

April 24th 7-10pm

- Similar format, (slightly) more questions.
- Cover everything. (evenly).
- No electronics allowed. 4-pages (double-sided)
- Office hours.
  - 21st, 23rd, afternoons (zoom).
  - TA: TBA.
  - Feel free to ask on Piazza.

Review: Poisson Process

- Poisson r.v.

Call $N \sim \mathrm{Poi}(\lambda)$ if $(\lambda > 0)$

$$\mathbb{P}(N=n) = e^{-\lambda}\frac{\lambda^n}{n!} \quad (n = 0, 1, 2, \ldots)$$

Fact: $X \sim \mathrm{Binom}(n, p)$. $p = \frac{\lambda}{n}$.

$$\mathbb{P}(X=k) = \frac{n!}{k!(n-k)!}\left(\frac{\lambda}{n}\right)^k \cdot \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\begin{pmatrix} \text{when } n \to +\infty \\ \lambda, k \text{ fixed} \end{pmatrix} = \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n^k}}_{\to 1} \cdot \frac{1}{k!} \lambda^k \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^{n-k}}_{\to e^{-\lambda}}$$

$$\to e^{-\lambda} \frac{\lambda^k}{k!}$$

(c.f. CLT in fixed-p setting).

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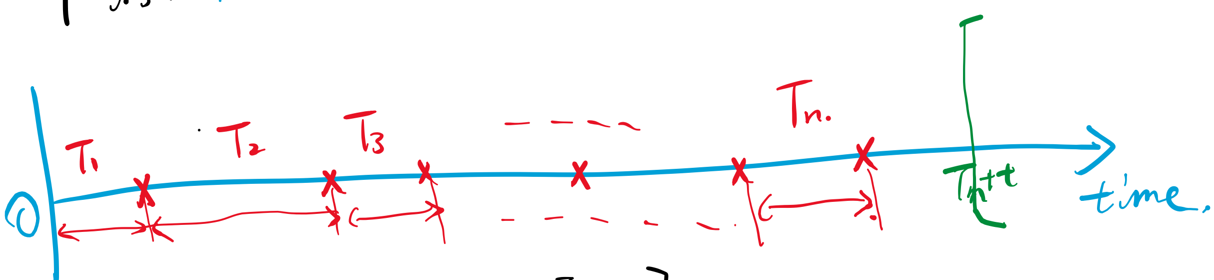
Poisson process

- First construction: marked-pt process.




$N(t) :=$ # marks in $[0, t]$.

$$T_1, T_2, \ldots, T_n, \ldots \overset{iid}{\sim} P.$$

We let
$$p_\lambda(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

(Exponential distribution).

Why $\exp(\lambda)$ distribution is special here?

Want to have Markov property.

$$\mathbb{P}(T > t+s \mid T > t) = \mathbb{P}(T > s).$$

(requirement by Markov property)

$$\mathbb{P}(T > t+s) = \mathbb{P}(T > t) \cdot \mathbb{P}(T > s)$$

Unique solution: $\mathbb{P}(T > t) = e^{-\lambda t}$.

Def. $PP(\lambda)$ is the marked point process where $\mathbb{P} = Exp(\lambda)$.

count in

Fact. $N(t) \sim Poi(\lambda t)$ $(\forall t)$.

Fact. $(N(t))_{t \geq 0}$ is Markov.

Independent Poisson increment. for $0 \leq s < t$.

$N(t) - N(s) \sim Poi(\lambda(t-s))$, and is independent of the past. $((N(\ell))_{0 \leq \ell \leq s})$.

(c.f. BM. independent normal increments)

Fact. $(N(t) - \lambda t)_{t \geq 0}$ is an MG.

Proof. $\mathbb{E}[N(t) \mid \mathcal{F}_s] = \mathbb{E}[N(t) - N(s) \mid \mathcal{F}_s] + N(s)$

$= \lambda(t-s) + N(s).$

Equivalent defn of $PP(\lambda)$.

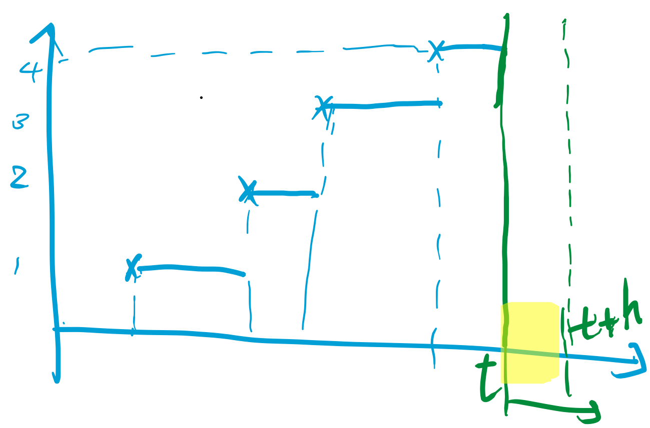
$(N(t))_{t\geq 0}$ taking value in $\mathbb{N}$; $N(0)=0$

$N(t)-N(s) \sim \mathrm{Poi}((t-s)\lambda)$, independent of the past.

Another equivalent characterization:

$(N(t))_{t\geq 0}$ taking non-neg int values, non-dec.

- $\mathbb{P}(N(t+h)-N(t)=1) = \lambda h + o(h)$.




- $\mathbb{P}(N(t+h)-N(t)\geq 2) = o(h)$

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Extension: time-inhomogeneous PP.

$$N(t)-N(s) \sim \mathrm{Poi}\left(\int_s^t \lambda(x)\,dx\right).$$ indep of the past

in other words

$$\mathbb{P}(N(t+h)-N(t)=1) = \lambda(t)\cdot h + o(h).$$

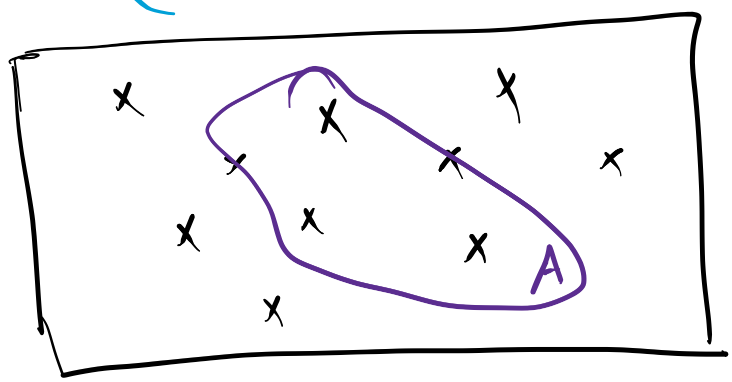
$$\left(\mathbb{P}(N(t+h)-N(t)\geq 2) = o(h)\right).$$

- Extension:

$$\mathbb{P}(N(t+h) - N(t) = 1 \mid N(t)) = \lambda(t, N(t)) \cdot h + o(h)$$

$$dN(t) = \lambda(t, N(t)) \cdot d\tilde{N}(t)$$

(where $\tilde{N}(t)$ is $PP(1)$).

- 



Poisson point process.

For any region $A$.

$N(A) = \#$ of pts inside $A$.

$$N(A) \sim \mathrm{Poi}\left(\int_A \lambda(x)\, dx\right)$$

For disjoint subsets $A, B$. $N(A) \perp N(B)$.

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Yet another equivalent characterization.

Construction procedure:

Fix $t > 0$, sample $N(t) \sim \mathrm{Poi}(\lambda t)$.

Conditioned on $N(t) = n$, we sample

$n$ marked points $\overset{iid}{\sim} \mathrm{Unif}([0, t])$

For $s < t$, let $N(s) :=$ # marked pts in $[0,s]$.

$(N(s))_{0 \le s \le t}$ is a PP.

This also generalizes to PPP.

- # pts in A. $N(A) \sim \text{Poi}\left(\int_A \lambda(x)\,dx\right)$.
- Conditionally on $N(A) = n$,
  marked pts are $n$ iid samples from $\dfrac{\lambda}{\int_A \lambda(x)\,dx}$.

Proof (1-dim, time-homogeneous case).

Consider # marks within interval $[a,b] \subseteq [0,t]$.

$N_{in} :=$ # marks inside $[a,b]$

$N_{out} :=$ # marks outside $[a,b]$.

Conditioned on $N(t) = N_{in} + N_{out} = n$.

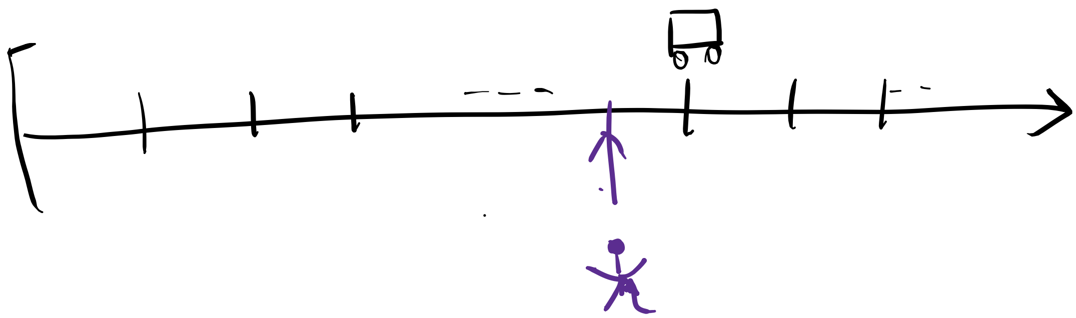
$$\mathbb{P}\left(N_{in} = k \,\middle|\, N(t) = n\right) = \frac{n!}{k!\,(n-k)!}\left(\frac{b-a}{t}\right)^k \cdot \left(\frac{t-b+a}{t}\right)^{n-k}.$$

($N_{in}$, $N_{out}$ are independent Poisson r.v.).

matching the prob where the marked pts are iid uniform samples.

e.g. "waiting time paradox".

"$\lambda$ buses per hour on average".



- If arriving every $1/\lambda$ hours.

  Wait time $\sim$ Uniform$\left(\left[0, \frac{1}{\lambda}\right]\right)$.

  $$\mathbb{E}[\text{wait time}] = \frac{1}{2\lambda}.$$

- If arriving according to $PP(\lambda)$.

  $$\mathbb{E}[\text{wait time}] = \mathbb{E}[\underbrace{\text{time till next inc}}_{\sim \text{Exp}(\lambda) \text{ indp of past.}}] = \frac{1}{\lambda}.$$

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"superposition property".

$(N_1(t))_{t \geq 0}$, $(N_2(t))_{t \geq 0}$ are indp PP w/ intensity $\lambda_1$, $\lambda_2$, then $(N_1(t) + N_2(t))_{t \geq 0}$ is also a PP w/ intensity param $\lambda_1 + \lambda_2$.

"Thining property".

$(N(t))_{t\geq 0} \sim PP(\lambda)$, suppose that for each marked point, indp label it with type $i$, w.p. $p_i$ (for $i=1,2,\cdots$)

For each $i \geq 1$, $N_i(t) :=$ # marked pts of type $i$ within $[0,t]$

are $PP(\lambda p_i)$, and indp of each other.

(Directly inferred from properties of Poisson r.v.'s)

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· Cts-time discrete-space MC's.

In general, state space $S$, init distr $\nu$.

Defn. $(X(t))_{t\geq 0}$ Markov if (for $0<t_1<t_2<\cdots<t_n$)

$$\mathbb{P}(X_0 = i_0, X_{t_1} = i_1, X_{t_2} = i_2, \cdots X_{t_n} = i_n)$$

$$= \nu_{i_0} \cdot P^{(t_1)}_{i_0, i_1} \cdot P^{(t_2 - t_1)}_{i_1, i_2} \cdot P^{(t_3-t_2)}_{i_2, i_3} \cdots P^{(t_n - t_{n-1})}_{i_{n-1}, i_n}.$$

where $\left(P^{(t)}_{ij}\right)_{\substack{i,j\in S\\ t\geq 0}}$ are transition probs.

eg. $PP(\lambda)$.

$$P_{ij}^{(t)} = \begin{cases} 0 & (j<i) \\ \dfrac{e^{-\lambda t} \cdot (\lambda t)^{j-i}}{(j-i)!} & (j \geq i) \end{cases}$$

"Generator of the process".

From now on, we assume $P_{ij}^{(t)}$ is a cts func of $t$ at $t=0$.

(Indeed, it is differentiable).

Def. $$g_{ij} := \lim_{t \to 0} \frac{P_{ij}^{(t)} - \delta_{ij}}{t} = \frac{d}{dt} P_{ij}^{(t)} \Big|_{t=0}.$$

where $\delta_{ij} = \mathbb{1}_{i=j}$.

We call $G := (g_{ij})_{i,j \in S}$ the generator of $(P_{ij}^{(t)})_{i,j \in S, t \geq 0}$.

eg. $PP(\lambda)$. $$g_{ij} = p_{ij}'(0) = \begin{cases} -\lambda & \text{when } i=j \\ \lambda & \text{when } j=i+1. \\ 0 & \text{otherwise.} \end{cases}$$

$$G = \begin{bmatrix} -\lambda & \lambda & & & 0 \\ & -\lambda & \lambda & & \\ & & -\lambda & \ddots & \\ & & & \ddots & \lambda \\ & 0 & & & -\lambda & \ddots \end{bmatrix}$$

- $g_{ii} = \lim_{t \to 0} \frac{p_{ii}^{(t)} - 1}{t} \le 0.$

- (for $j \neq i$). $g_{ij} = \lim_{t \to 0} \frac{p_{ij}^{(t)}}{t} \ge 0.$

- $\sum_{j \in S} g_{ij} \underset{\uparrow}{=} \lim_{t \to 0} \frac{\sum_{j \in S} p_{ij}^{(t)} - 1}{t} = 0.$

can be justified

Indeed, $G$ contains all the info about $(p_{ij}^{(t)})_{\substack{t \ge 0 \\ i,j \in S}}$.

Thm. If $G$ is the generator, then

$$P^{(t)} \left( := \left( p_{ij}^{(t)} \right)_{i,j \in S} \right) = \exp(tG).$$

Matrix exponential.

$$\exp(tG) = I + tG + \frac{t^2 G^2}{2!} + \frac{t^3 G^3}{3!} + \cdots\cdots$$

Additional remark: computing matrix exp

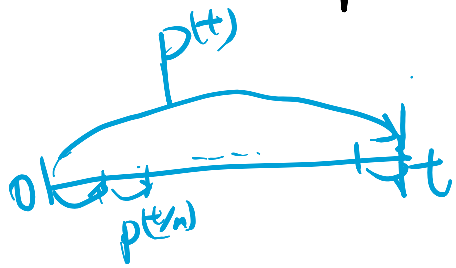
If $G = P \Lambda P^{-1}$

$$\exp(tG) = P \left( \sum_{n=0}^{+\infty} \frac{(t\Lambda)^n}{n!} \right) P^{-1}$$

$$= P \cdot \mathrm{diag}\left( e^{t\lambda_1}, e^{t\lambda_2}, \cdots \right) P^{-1}$$

where $\lambda_1, \lambda_2 \cdots$ are eigenvalues

Proof sketch: $P^{(t)} = \left(P^{(t/n)}\right)^n$.




$$= \lim_{n\to+\infty} \left(P^{(t/n)}\right)^n$$

$$= \lim_{n\to+\infty} \left(I + \frac{t}{n} G + o\left(\frac{1}{n}\right)\right)^n$$

$$= \exp(tG).$$

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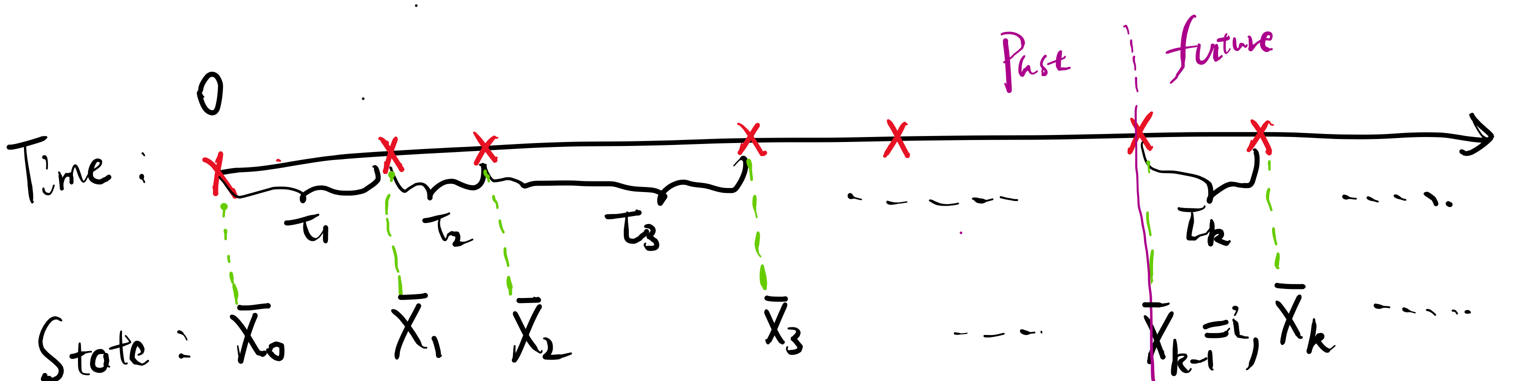
What does a CTMC look like?

Given a generator $G = (g_{ij})_{i,j\in S}$.




$$X_t = \bar{X}_{k-1} \quad \text{for } t \in \left[\sum_{i=1}^{k-1} \tau_i,\ \sum_{i=1}^{k} \tau_i\right).$$

Conditioned on everything up to time $\sum_{i=1}^{k-1} \tau_i$.

we sample

$$\tau_k \begin{cases} \sim \operatorname{Exp}(-g_{ii}). & \text{when } g_{ii} < 0 \\ \text{stay at } \bar{X}_{k-1} \text{ forever.} & \text{when } g_{ii} = 0 \end{cases}$$

(when $\bar{X}_{k-1} = i$)

And the next step transition

$$\bar{X}_k \sim \frac{g_{i,j}}{-g_{i,i}} = \frac{g_{i,j}}{\sum_{l \neq i} g_{i,l}}.$$

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e.g. Stationary distribution $\pi$

$$\pi G = 0$$

e.g. reversible $\Longleftrightarrow$ $\pi_i g_{ij} = \pi_j g_{ji}$

Thm. Irreducible $(P^{(t)})_{t \geq 0}$, w/ stationary distr $\pi$.

then $\lim_{t \to +\infty} P_{ij}^{(t)} = \pi_j \quad (\forall i, j \in S)$.

No need to worry about periodicity

— $P_{ii}^{(t)} > 0$ (when $t$ is small enough).