STA 447/2006 Lecture 2

From last time.

State i recurrent $\Longleftrightarrow \sum_{n=1}^{+\infty} p_{ii}^{(n)} = +\infty$.

Detour. In general Borel-Cantelli Lemma.

Let $(E_n)_{n=1}^{+\infty}$ be a collection of events.

(eg $E_n = \{X_n = i\}$)

If $\sum_{n=1}^{+\infty} \mathbb{P}(E_n) < +\infty$, then $\{E_n\}_{n=1}^{+\infty}$ only happens finite many times. w.p. 1

Proof. $\sum_{n=1}^{+\infty} \mathbb{P}(E_i) = \mathbb{E}\left[\sum_{n=1}^{+\infty} \mathbb{1}_{E_n}\right]$

$= \mathbb{E}[\# \text{ of } (E_n)_{n=1}^{+\infty} \text{ that happens}]$

Backward implication not true in general

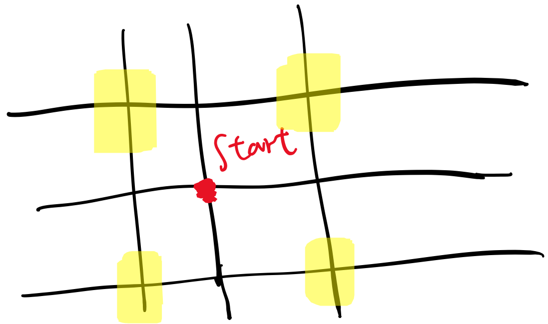
In MC, geometric distribution has finite expectation.

SRW

— 1D.

$p_{00}^{(n)} \sim \frac{1}{\sqrt{n}}$ $\sum_{n=1}^{+\infty} p_{00}^{(n)} = +\infty$

— 2D, 3D, ... ?



$$P_{(i_1, i_2, \cdots i_d),(j_1, j_2, \cdots j_d)} = \begin{cases} 2^{-d} & \text{when } |i_k - j_k| = 1 \text{ for each } k \\ 0 & \text{otherwise} \end{cases}$$

$$(X_n)_{n=1}^{+\infty} = \left[\left(X_n^{(1)}, X_n^{(2)}, \cdots, X_n^{(d)}\right)\right]_{n=1}^{+\infty}$$

Each $X_n^{(i)}$ is 1-D SRW, for $i=1,2,\cdots d$.

and independent w/ each other.

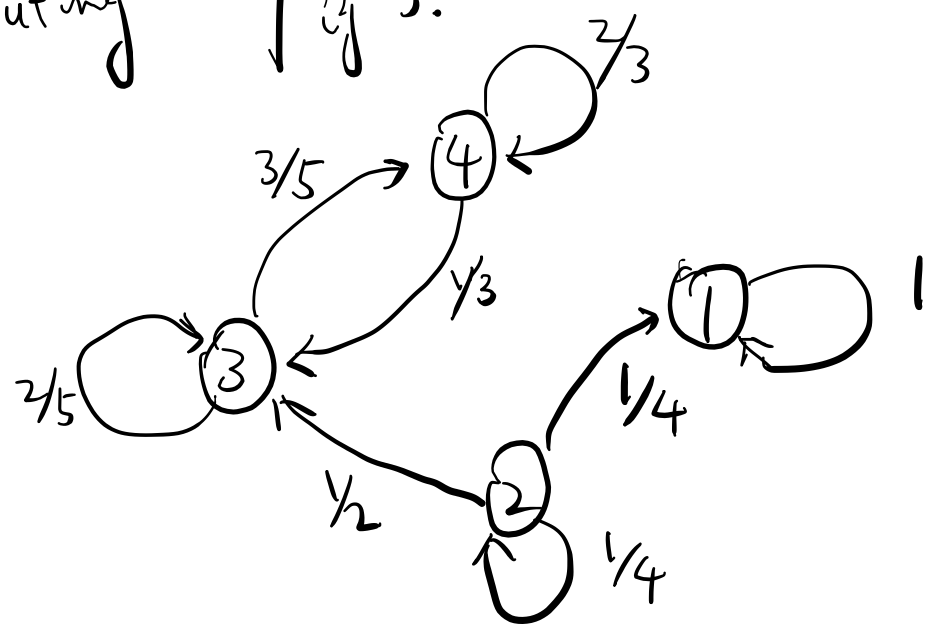
$$P_{00}^{(n)} = \left(2^{-n}\binom{n}{n/2}\right)^d \sim n^{-d/2}.$$

$$\sum_{n=1}^{+\infty} P_{00}^{(n)} \begin{cases} = +\infty & (d=1,2) \\ < +\infty & (d \geq 3). \end{cases}$$

A drunk man will get home, while a drunk bird may not!

Computing f_{ij}'s.

e.g.



$f_{11} = 1$, $f_{1j} = 0$ (for $j = 2, 3, 4$).

$f_{22} = 1/4$.

$f_{21} = \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4} + \left(\frac{1}{4}\right)^2 \frac{1}{4} + \cdots\cdots = 1/3$.

$f_{23} = \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} + \left(\frac{1}{4}\right)^2 \frac{1}{2} + \cdots\cdots = \frac{2}{3}$.

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f - expansion.

$f_{ij} = \mathbb{P}_i(\text{even visit } j)$.

$= \sum_{k \in S} \mathbb{P}_i(X_1 = k) \cdot \mathbb{P}_k(\text{even visit } j)$

$\left(= \cancel{\sum_{k \in S} P_{ik} f_{kj}} \quad \text{incorrect} \right)$

$$= P_{ij} + \sum_{\substack{k \in S \\ k \neq j}} P_{ik} \cdot f_{kj}.$$

($|S|^2$ variables, $|S|^2$ equations)

eg Gambler's ruin.

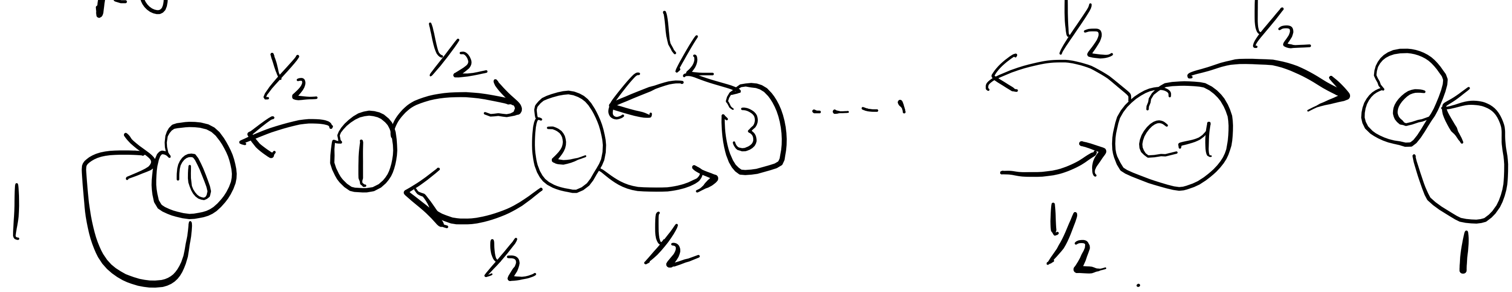
- Initial money $a \in \mathbb{N}_+$
- $\begin{cases} \text{win } \$1 & \text{w.p } 1/2 \\ \text{lose } \$1 & \text{w.p } 1/2. \end{cases}$
- Stop when the wealth reaches $c \in \mathbb{N}_+$ $(c > a.)$

  also when the wealth becomes 0.

$X_t :=$ amount of money in round t

$S = \{0, 1, 2, \cdots, c\}$

$X_0 = a$.



Interested in

$\mathbb{P}_a($ever visiting $c) \ \forall a/c.$

$\|$

f_{ac}

$f_{00} = 1, \quad f_{cc} = 1.$ (0 and c recurrent)

$f_{ii} < 1$ when $i \notin \{0, c\}$ (transient states)

$i \in \{1, 2, \cdots c-1\}$

By f-expansion.

$$f_{ic} = P_{ic} + \sum_{\substack{k \in S \\ k \neq c}} P_{ik} f_{kc}$$

$$\left(= \sum_{k \in S} P_{ik} f_{kc} \quad \text{since } f_{cc} = 1 \right)$$

$$= \frac{1}{2} f_{(i+1)c} + \frac{1}{2} f_{(i-1)c}$$

$$f_{(i+1)c} - f_{ic} = f_{ic} - f_{(i-1)c}$$

(for each $i \in \{1, 2, \cdots c-1\}$).

$$1 = f_{cc} - f_{0c} = \sum_{i=1}^{c} \left(f_{ic} - f_{(i-1)c} \right)$$

$$= c \cdot \left(f_{1c} - f_{0c} \right).$$

Conclusion $f_{ac} = a/c$.

Extension $p \neq 1/2$.

$$p_{i(i+1)} = p, \quad p_{i(i-1)} = 1-p$$

for each $i \in \{1, 2, \cdots, c-1\}$.

f -expansion

$$f_{ic} = p f_{(i+1)c} + (1-p) f_{(i-1)c}.$$

$$f_{(i+1)c} - f_{ic} = \frac{1-p}{p} \cdot \left(f_{ic} - f_{(i-1)c}\right).$$

$\left(f_{(i+1)c} - f_{ic}\right)_{i=0}^{c-1}$ is a geometric seq.

$$f_{(i+1)c} - f_{ic} = \left(\frac{1-p}{p}\right)^{i} \cdot \left(f_{1c} - f_{0c}\right).$$

We also know

$$1 = f_{cc} - f_{0c} = \sum_{i=0}^{c-1} \left(f_{(i+1)c} - f_{ic}\right) = \left(f_{1c} - f_{0c}\right) \cdot \sum_{i=0}^{c-1} \left(\frac{1-p}{p}\right)^{i}$$

$$f_{ac} = \frac{\left(\frac{1-p}{p}\right)^{a} - 1}{\left(\frac{1-p}{p}\right)^{c} - 1}.$$

Def. Say i communicates to j $(i \to j)$

if $f_{ij} > 0$.

Notation "$i \leftrightarrow j$" to denote

$i \to j$ and $j \to i$.

Def. irreducible MC iff

for any $i, j \in S$, $i \longleftrightarrow j$.

(If not, what it means to be "reduced").

e.g. SRW 1-D irreducible.

Higher-dim: reducible (by our def.)

irreducible (pick random coordinate to move at each step).

e.g. Frog walk irreducible.

e.g. Gambler's ruin. reducible.

Fact: If $i \longleftrightarrow j$, then i recurrent

$\Updownarrow$

j recurrent.

Corollary. "case theorem".

If MC is irreducible, then one of 2 cases.

(i) All states recurrent "recurrent MC"

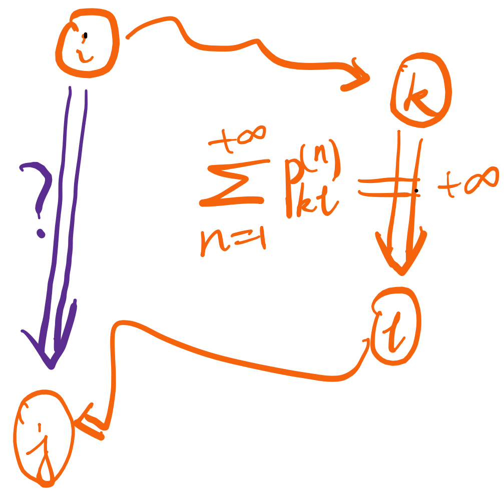
(ii) All states transient. "transient MC".

e.g. SRW (1-D). All states are recurrent (Also applicable to $d>1$ case)

Proof of fact

"sum lemma".

If $i \to k$ and $l \to j$



If $\sum_{n=1}^{+\infty} p_{kl}^{(n)} = +\infty$

then $\sum_{n=1}^{+\infty} p_{ij}^{(n)} = +\infty$.

Proof of sum lemma.

Since $i \to k$, $\exists\ m>0$, $p_{ik}^{(m)} > 0$

$l \to j$, $\exists\ r>0$, $p_{lj}^{(r)} > 0$.

(when $n > m+r$)

$$p_{ij}^{(n)} \geq p_{ik}^{(m)} \cdot p_{kl}^{(n-m-r)} \cdot p_{lj}^{(r)}$$

$$\sum_{n=1}^{+\infty} p_{ij}^{(n)} \geq \underbrace{\boxed{p_{ik}^{(m)} \cdot p_{lj}^{(r)}}}_{>0} \cdot \underbrace{\sum_{t=1}^{+\infty} p_{kl}^{(t)}}_{=+\infty} = +\infty.$$

From sum lemma to the fact about recurrence.

We let $j=i$, $l=k$ in sum lemma.

sum lemma becomes:

If $i \to k$, $k \to i$, then $\boxed{\sum_{n=1}^{+\infty} P_{kk}^{(n)} = +\infty}$ $\Leftrightarrow$ k is rec

implies that $\boxed{\sum_{n=1}^{+\infty} P_{ii}^{(n)} = +\infty}$ $\Leftrightarrow$ i is rec.

By recurrent state thm, we conclude the proof.

Also,

$$\sum_{n=1}^{+\infty} P_{ii}^{(n)} \begin{cases} <+\infty & \text{(transient)} \\ =+\infty & \text{(recurrent)} \end{cases}$$

now about

$$\sum_{n=1}^{+\infty} P_{ij}^{(n)} \ ?$$

Fact: $\boxed{\begin{array}{l}\text{Recurrent} \\ \\ \text{Transient}\end{array}}$

Recurrent MC. $\sum_{n=1}^{+\infty} P_{ij}^{(n)} = +\infty$ $(\forall\, i,j \in S)$

Transient MC $\sum_{n=1}^{+\infty} P_{ij}^{(n)} < +\infty$ $(\forall\, i,j \in S)$.

When we call it that way, we implicitly assume irreducibility.

Proof. $\sum_{n=1}^{+\infty} P_{ij}^{(n)} = \mathbb{E}_i[\#\text{visits to } j]$.

(Recurrent.) $\geq P_{ij}^{(m)} \cdot \mathbb{E}_j[\#\text{visits to } j] = +\infty$

(Transient) $= \dfrac{f_{ij}}{1-f_{jj}} < +\infty$ (Geometric r.v.).

Special case: finite state space

Thm. If $|S| < +\infty$, irreducible, then recurrent.

Proof. Fix $i \in S$,

$$\sum_{j \in S} \boxed{\sum_{n=1}^{+\infty} P_{ij}^{(n)}} = \sum_{n=1}^{+\infty} \left(\sum_{j \in S} P_{ij}^{(n)} \right) = \sum_{n=1}^{+\infty} 1 = +\infty$$

finite summation.

So $\exists j \in S$, $\sum_{n=1}^{+\infty} P_{ij}^{(n)} = +\infty$.

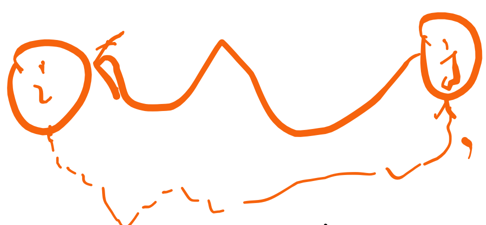
By the fact above, this implies recurrence.

Goal: necessary and sufficient cond for an irreducible MC to be recurrent or transient.

"f-Lemma". If $j \to i$ and $f_{jj} = 1$

then $f_{ij} = 1$.

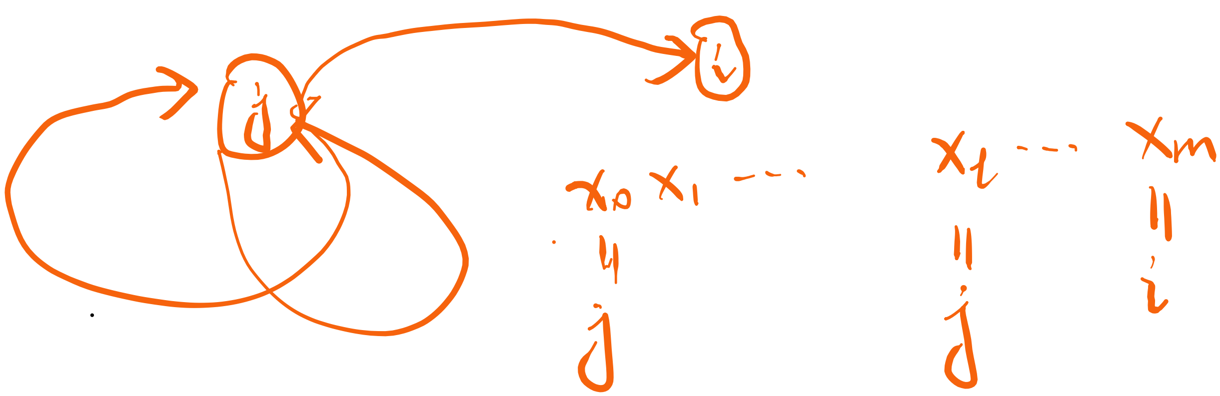
(info about f_{ij} for $i \neq j$).



Technical tool: "Hit Lemma". (true for any i, j).

$H_{ij} = \{$MC hits i before returning to $j\}$

If $j \to i$ then $\mathbb{P}_j(H_{ij}) > 0$.



Then we know

$$p_{x_l x_{l+1}} p_{x_{l+1} x_{l+2}} \cdots p_{x_{m-1} x_m} > 0.$$

corresponds to a path that hits i before returning to j.

Back to the proof of f-lemma.

By hit lemma, since $j \to i$

$$\mathbb{P}_j(H_{ij}) > 0.$$

j recurrent

$$0 = \mathbb{P}_j(\text{never return to } j)$$

$$\geq \underbrace{\mathbb{P}_j(H_{ij})}_{>0} \cdot \underbrace{\mathbb{P}_i(\text{never visit to } j)}_{=0}.$$

By strong Markov, and that trajectories that visit i and never come back is a subset of trajectories that never come back.

So $f_{ij} = 1$.

"Infinite returns lemma".

For irreducible MC.

$$\begin{cases} \text{Recurrent} & \text{then } \forall i,j \in S, \quad \mathbb{P}_i(N(j)=+\infty)=1. \\ \text{Transient} & \text{then } \forall i,j \in S \quad \mathbb{P}_i(N(j)=+\infty)=0. \end{cases}$$

Proof. Recurrent case:

$$\boxed{f_{jj}=1 \quad j \to i \quad \underset{f\text{-lemma}}{\Longrightarrow} \quad f_{ij}=1.}$$

(By last lecture)

The only new ingredient.

$\forall k \in \mathbb{N}$,

$$\mathbb{P}_i(N(j) \geq k) = f_{ij} \cdot f_{jj}^{k-1} = 1.$$

So $\mathbb{P}_i(N(j)=+\infty)=1.$

Transient case:

$$\mathbb{E}_i[N(j)] = \frac{f_{ij}}{1-f_{jj}} < +\infty$$

So $N(j) < +\infty$, w.p. 1.

"Big thm" Recurrence Equivalence thm.

If MC is irreducible, the following are equivalent.

(i) $\exists\, k, l \in S$, s.t. $\sum_{n=1}^{+\infty} p_{kl}^{(n)} = +\infty$

(ii) $\forall\, k, l \in S$, $\sum_{n=1}^{+\infty} p_{kl}^{(n)} = +\infty$

(iii) $\exists\, k \in S$ $f_{kk} = 1$

(iv) $\forall\, k \in S$ $f_{kk} = 1$.

(v) $\forall\, i, j \in S$ $f_{ij} = 1$

(vi) $\exists\, k, l \in S$. $\mathbb{P}_k\left(N(l) = +\infty\right) = 1$.

(vii) $\forall\, k, l \in S$ $\mathbb{P}_k\left(N(l) = +\infty\right) = 1$.

(i)(ii) by recurrent state thm and sum lemma.

(iii)(iv) simultaneously recurrent

(v) f - lemma

(vi) (vii) Infinite visits lemma.

The "$\exists$" version of (v)

is not equiv to recurrence.

e.g. $X_{n+1} = X_n + \varepsilon_{n+1}$ $(\forall n)$

where $\varepsilon_{n+1} = \begin{cases} 1 & \text{w.p. } 2/3, \\ -1 & \text{w.p. } 1/3. \end{cases}$

$f_{01} = 1$, but transient.