STA 447/2006 Lecture 3.

Irreducible MC.

$\forall$ or $\exists$ $i,j \in S$

$$\mathbb{P}_i(N(j)=+\infty)=1$$

$$\Updownarrow$$

$\forall$ or $\exists$ i, $f_{ii}=1$ $\Longleftrightarrow$ Recurrence $\Longleftrightarrow$ $\forall$ or $\exists$ $i,j \in S$, $\sum_{n=1}^{+\infty} P_{ij}^{(n)} = +\infty$

$$\Updownarrow$$

$$\forall i,j \in S,\ f_{ij}=1$$

$\boxed{\exists i,j,\ f_{ij}=1}$ is not an equivalent cond.

Prop. $\exists$ an irreducible MC s.t. transient but $f_{ij}=1$ for some i,j.

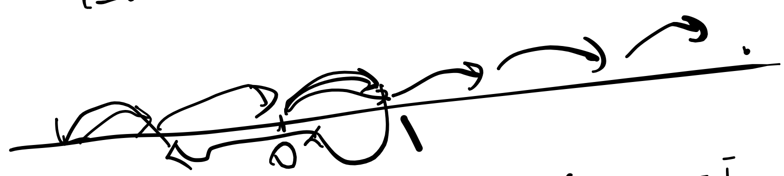
Proof. $X_0=0$ $\quad X_{n+1}=X_n+\varepsilon_{n+1}$

where $\varepsilon_{n+1}=\begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } 1-p \end{cases}$ where $p>\frac{1}{2}$ $(p<1)$.

- irreducible ✓
- Transience?

$$P_{00}^{(n)}\begin{cases} 0 & (n \text{ is odd}) \\ \binom{n}{n/2} p^{n/2}(1-p)^{n/2} \approx (4p(1-p))^{n/2}\cdot\sqrt{\frac{2}{\pi n}} \end{cases}$$

$$\sum_{n=0}^{+\infty} p_{00}^{(n)} \leq \sum_{n=0}^{+\infty} (4p(1-p))^{n/2} < +\infty.$$



$\frac{1}{n} X_n = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i$

By LLN, $\frac{1}{n} X_n \xrightarrow{a.s.} \mathbb{E}[\varepsilon_1] = 2p-1 > 0.$

which implies $\mathbb{P}_0(X_n \to +\infty) = 1.$

So $\mathbb{P}_0(\text{hit } 1) = 1 \quad f_{01} = 1.$

Similarly, "transience equivalence thm"

$\forall$ or $\exists$ $i,j \in S$

$\mathbb{P}_i(N(j) = +\infty) = 0$

$N(j)$ is $\begin{cases} +\infty & \text{w.p. } 1 \\ \text{geometric distribution} \end{cases}$

$\Updownarrow$

$\forall$ or $\exists$ $i \in S$ $\qquad\qquad\qquad\qquad$ $\forall$ or $\exists$ $i,j \in S$

$f_{ii} < 1 \iff \text{Transience} \iff \sum_{n=0}^{+\infty} p_{ij}^{(n)} < +\infty.$

$\Updownarrow$

$\exists\, i,j.\ f_{ij} < 1.$

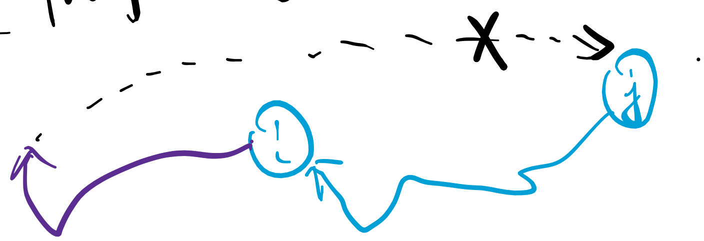
Reducible MC.

(Finite statespace). $P = \left[\begin{array}{c|c} \tilde{P} & 0 \\ \hline S & Q \end{array}\right] \begin{array}{l} \}\ \text{recurrent} \\ \}\ \text{Transient} \end{array}$

Fact : i transient and j recurrent

then $j \not\to i$.

(Proof : similar to f-lemma)



$$P^n = \left[\begin{array}{c|c} \tilde{P}^n & 0 \\ \hline S_n & Q^n \end{array}\right]$$

$Q^n \to 0$ since these state are transient.

$$\tilde{P} = \left[\begin{array}{c|c|c|c} \tilde{P}_1 & 0 & \cdots & 0 \\ \hline 0 & \tilde{P}_2 & & \\ \hline & & \ddots & \\ \hline 0 & \cdots & & \tilde{P}_r \end{array}\right]$$

eg. can compute $\mathbb{P}_i$(ends up in group k)

(cf-expansion, similar to gambler's ruin) for $i \in$ Transient states

k is one of the recurrent groups.

Main question

$$\lim_{n\to+\infty} \mathbb{P}_i(X_n = j) \quad ?$$

Suppose $p_{ij}^{(n)} \to q_j \quad (\forall j)$.

then $p_{ij}^{(n+1)} \to q_j$

$$p_{ij}^{(n+1)} = \sum_{k\in S} p_{ik}^{(n)} \cdot p_{kj}$$

$$q_j = \sum_{k\in S} q_k \cdot p_{kj}. \qquad q = qP$$

Def. We say π is stationary for P (π is a probability distribution).

when $\pi = \pi P$

π : $1\times S$ vector

(in general, you may get π s.t. $\sum_x \pi(x) = +\infty$, stationary measure).

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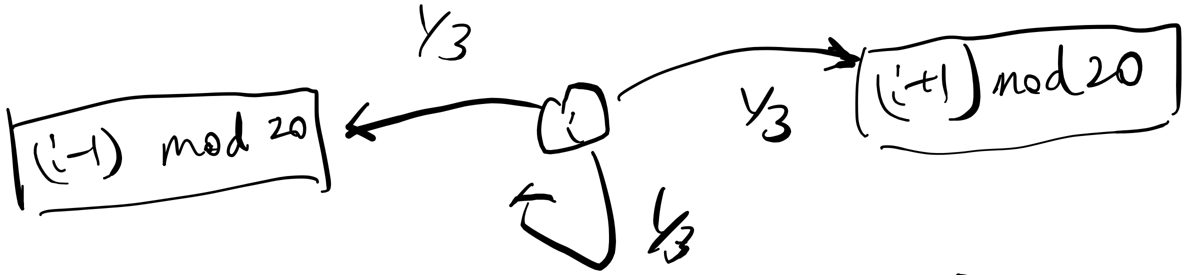
eg Frog walk 20 states.



Guess:

$$\pi = \left[\frac{1}{20}, \frac{1}{20}, \cdots, \frac{1}{20}\right].$$

$\forall i \in S$

$$\frac{1}{20} = \pi_i \overset{?}{=} \sum_j \pi_j P_{ji} = \frac{1}{20} \cdot \frac{1}{3} \cdot 3 = \frac{1}{20}$$

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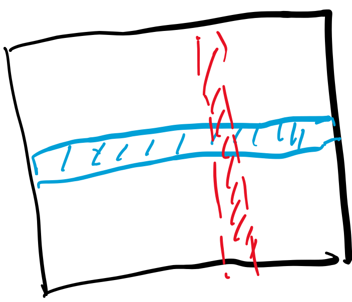
eg. "Doubly stochastic matrices".

$P =$  ← transition prob from state i.

$$\sum_{j \in S} P_{ij} = 1.$$

If $\sum_{i \in S} P_{ij} = 1 \quad (\forall j \in S)$ then we call the transition matrix P "doubly stochastic".

Fact. If P is doubly stochastic, then uniform distribution $\pi = \left[\frac{1}{|S|}, \frac{1}{|S|}, \cdots, \frac{1}{|S|}\right]$ is stationary.

Proof.

$$\frac{1}{|S|} = \pi_i \stackrel{?}{=} \sum_{j\in S} \pi_j P_{ji} = \sum_{j \in S} \frac{1}{|S|} \cdot P_{ji} = \frac{1}{|S|}$$

(Converse also true).

Def. Call it Reversible / detailed balance w.r.t. π.

if $\pi_i P_{ij} = \pi_j P_{ji}$ for $\forall\, i,j \in S$.

Fact. P is reversible w.r.t. π then π is a stationary distribution of P.

Proof.

$$\pi_j \stackrel{?}{=} \sum_i \pi_i \cdot P_{ij} = \sum_i \pi_j P_{ji} = \pi_j$$

e.g. Frog walk.

e.g. Ehrenfest Urn.

Box 1 Box 2

Each time:

- Randomly pick a ball
- Put it to opposite side.

$X_n = \#$ balls in Box 1 at n-th round.

$$P_{i(i+1)} = \frac{d-i}{d} \qquad P_{i(i-1)} = \frac{i}{d} \quad (\forall i).$$

$$\pi_i = \binom{d}{i} \cdot 2^{-d} \qquad (\mathrm{Binom}(d, 1/2))$$

Detailed balance condition:

$$\pi_i P_{i(i+1)} = 2^{-d} \binom{d}{i} \cdot \frac{d-i}{d}$$

$$= 2^{-d} \cdot \frac{d!}{i!(d-i)!} \cdot \frac{d-i}{d} = 2^{-d} \cdot \frac{(d-1)!}{i!\,(d-1-i)!}$$

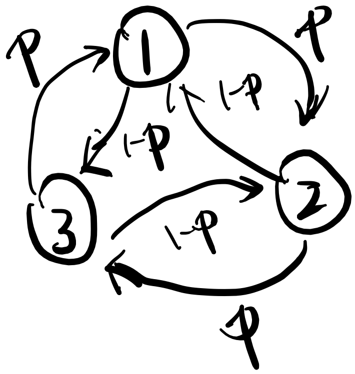
$$\pi_{i+1} P_{(i+1)i} = 2^{-d} \cdot \binom{d}{i+1} \cdot \frac{i+1}{d}$$

$$= 2^{-d} \cdot \frac{d!}{(i+1)!(d-i-1)!} \cdot \frac{i+1}{d} = 2^{-d} \frac{(d-1)!}{i!\,(d-1-i)!}$$

We can also solve it directly.
($d+1$ variables, $d+1$ eqs, solution always exists).

Prop. $\exists$ a MC P w/ stationary distribution π
s.t. P is not reversible w.r.t. π.

eg.



$\pi_i = \frac{1}{3} \quad \forall i \in S.$

$$\frac{1}{3} = \pi_2 \overset{?}{=} \sum_{j \in S} P_{j2}\,\pi_j = \frac{1}{3}\cdot(1-p) + \frac{1}{3}\cdot p = \frac{1}{3}.$$

But detailed Balance is false (when $p \neq \frac{1}{2}$).

$$\frac{p}{3} = \pi_1 P_{12} \overset{?}{=} \pi_2 P_{21} = \frac{1-p}{3}$$

Existence & Uniqueness of stationary distribution.

eg. SRW. $\pi_i = 1$ for each $i \in S$ is a stationary measure, but not stationary distribution

eg. Reducible MC.

$$\begin{array}{c} \pi_1 \\ \pi_2 \end{array} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

$\forall \lambda \in [0,1]$

$[\lambda \pi_1 \quad (1-\lambda)\pi_2]$ is a stationary distribution.

e.g. Finite state space. Stationary distribution always exists

(Brower fixed-pt thm).

Thm ("vanishing probabilities").

If for any $i,j \in S$, $\lim_{n\to+\infty} p_{ij}^{(n)} = 0$

then the MC does not have a stationary distribution.

Proof. Suppose $\exists$ a stationary distribution π.

$$\pi = \pi P = \pi P^2 = \cdots = \pi P^n$$

"A physics proof"

$$\pi_j = \lim_{n\to+\infty} \sum_{i\in S} \pi_i \, p_{ij}^{(n)} = \sum_{i\in S} \pi_i \lim_{n\to+\infty} p_{ij}^{(n)} = 0$$

For rigorous proof, need to justify $\lim \sum = \sum \lim$.

"M-test": $\{x_{n,k}\}_{n,k\in\mathbb{N}}$

Suppose that $\forall k$, $\lim_{n\to+\infty} x_{n,k}$ exists,

and $\sum_{k=1}^{+\infty} \sup_{n\geq 1} |x_{n,k}| < +\infty$

then $\lim_{n\to+\infty} \sum_{k=1}^{+\infty} x_{n,k} = \sum_{k=1}^{+\infty} \lim_{n\to+\infty} x_{n,k}$

(Proof. Appendix A.11.1 of Rosenthal)

$$\sum_{i\in S} \sup_{n\geq 1} \left| \pi_i P_{ij}^{(n)} \right| .$$

$$\leq \sum_{i\in S} \pi_i = 1.$$

$$\pi_j = \lim_{n\to+\infty} \sum_{i\in S} P_{ij}^{(n)} \pi_i$$

$$= \sum_{i\in S} \left(\lim_{n\to+\infty} P_{ij}^{(n)} \right) \cdot \pi_i$$

$$= 0.$$ Contradiction.

Can we relax the "$\forall i,j$" condition?

"Vanishing Lemma".

If for some $k,l\in S$ $\lim_{n\to+\infty} P_{kl}^{(n)} = 0$

and $k\to i$, and $j\to l$ for some $i,j\in S$,

then $\lim_{n\to+\infty} P_{ij}^{(n)} = 0$.

Cor. For an irreducible MC, if $\exists k, l \in S$, $\lim_{n\to+\infty} P_{kl}^{(n)} = 0$ then there is no stationary distribution.

Cor. A transient, irreducible MC doesn't have a stationary distribution.

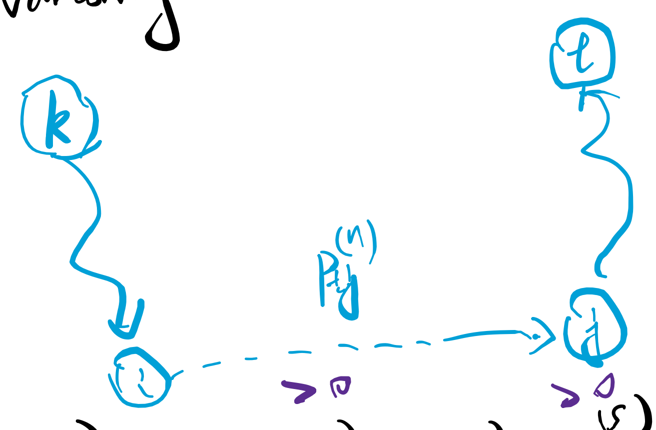
(Transience $\Leftrightarrow \sum_{n=1}^{+\infty} P_{ij}^{(n)} < +\infty \Rightarrow P_{ij}^{(n)} \to 0$).

Cor "vanishing together".
For irreducible MC, either (i) $\lim_{n\to+\infty} P_{ij}^{(n)} = 0$ $(\forall i, j \in S)$

or (ii) $\lim_{n\to+\infty} P_{ij}^{(n)} \neq 0$ $(\forall i, j \in S)$.

(the latter case doesn't necessarily mean convergence).

Proof of vanishing lemma.



$\exists r, s \geq 0$

$P_{ki}^{(r)} > 0$, $P_{jl}^{(s)} > 0$.

$\left(\begin{array}{l} r, s \text{ fixed} \\ \text{let } n \to +\infty \end{array} \right)$

$$\underbrace{P_{kl}^{(r+n+s)}}_{\to 0 \text{ (by assumption)}} \geq \underbrace{P_{ki}^{(r)}}_{} \cdot P_{ij}^{(n)} \cdot \underbrace{P_{jl}^{(s)}}_{}$$

So $P_{ij}^{(n)} \to 0$.

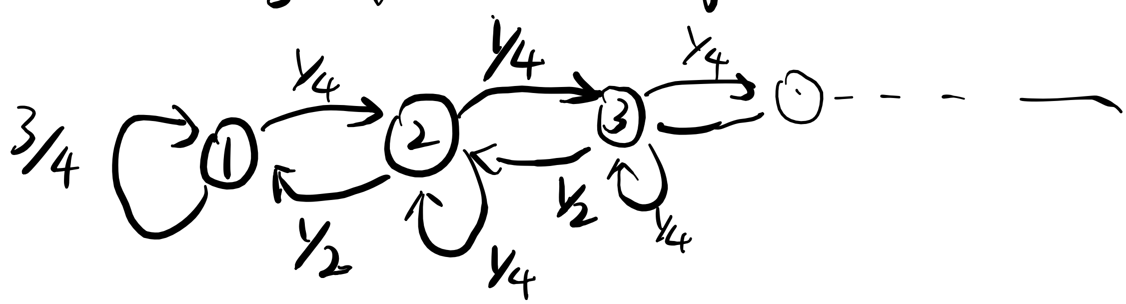
eg. 1-dim SRW

$$p_{00}^{(n)} \approx c \cdot \frac{1}{\sqrt{n}} \to 0 \quad (\text{as } n \to +\infty)$$

So stationary distribution doesn't exist.

eg. $\exists$ an infinite statespace MC, s.t. stationary exists.

$$S = \{1, 2, 3, \cdots\}.$$



$(i \geq 2)$ $\quad P_{ii} = \frac{1}{4}, \quad P_{i(i+1)} = \frac{1}{4}, \quad P_{i(i-1)} = \frac{1}{2}$

$\pi_i = 2^{-i}$. reversible: $\pi_i P_{i(i+1)} = 2^{-(i+2)} = \pi_{i+1} \cdot P_{(i+1)i}$.

Non-convergence failure modes:

eg.  $\forall \lambda \in [0,1]$ $[\lambda, 1-\lambda]$ is stationary. (eg. $[\frac{1}{2}, \frac{1}{2}]$).

But if starting from 1, impossible to have $\lim_{n \to +\infty} P_{12}^{(n)} = \frac{1}{2}$

To rule out: irreducibility.

e.g.

Unique stationary:

$\left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$.

In general, we may have 3 subsets of S w/ cyclic behavior

$$p_{11}^{(n)} = \begin{cases} 0 & \text{when } \frac{n}{3} \notin \mathbb{Z} \\ 1 & \text{when } \frac{n}{3} \in \mathbb{Z}. \end{cases}$$

In order to rule out: aperiodicity.

Def. Period of a state $i \in S$ is greatest common divisor (gcd) of the set $\{n \geq 1 : p_{ii}^{(n)} > 0\}$

e.g. for the cyclic chain above the set is $\{3, 6, 9, \cdots\}$, period is 3.

e.g. If $p_{ii} > 0$, then period of i is 1.

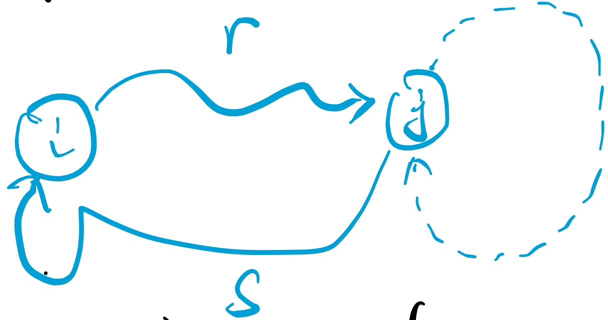
e.g. If $p_{ii}^{(n)} > 0$, and $p_{ii}^{(n+1)} > 0$, then the period of i is also 1.

Lemma (Equal period). If $i \leftrightarrow j$ then i, j has the same period.

Corollary. If the chain is irreducible, then all states have the same period.

Def. (Aperiodicity) period $=1$ for a state on a MC (in the irreducible case).

Proof of Equal period lemma.



Suppose $P_{jj}^{(n)} > 0$ for some n

then $P_{ii}^{(r+n+s)} \geq P_{ij}^{(r)} \cdot P_{jj}^{(n)} \cdot P_{ji}^{(s)} > 0$.

Additionally, $P_{ii}^{(r+s)} \geq P_{ij}^{(r)} \cdot P_{ji}^{(s)} > 0$.

Suppose t_i : period of i

t_j : period of j

$\frac{r+n+s}{t_i} \in \mathbb{Z}$, $\frac{r+s}{t_i} \in \mathbb{Z} \Longrightarrow \frac{n}{t_i} \in \mathbb{Z}$

t_i is a common divisor of $\{n \geq 1 : P_{jj}^{(n)} > 0\}$

$t_i \leq t_j$

By symmetry $t_j \leq t_i$ so $t_i = t_j$.

Thm (MC convergence)

If a MC is irreducible, aperiodic and has a stationary distribution π, then $\forall i,j \in S$

$$\lim_{n\to+\infty} P_{ij}^{(n)} = \pi_j$$ (starting from fixed i)

and, for $X_0 \sim \nu$

$$\lim_{n\to+\infty} \mathbb{P}(X_n = j) = \pi_j .$$

(automatically implies uniqueness).