STA 447/2006 Lecture 4

Midterm #1 Next week

- Same location
- 6:10pm- 8pm
- 2 pages of cheatsheet allowed
  electronics (incl. calculator) not allowed
  (and will not be useful).
- Covers everything in the first 4 weeks.

Thm (MC convergence)

If P is irreducible, aperiodic,
and has stationary distribution
then $\forall i, j \in S$, $\lim_{n\to+\infty} P_{ij}^{(n)} = \pi_j$.

Proof idea:

Main lemma. (Markov forgetting) (under same assumptions).

$\forall i, j, k \in S$, $\lim_{n\to+\infty} \left| P_{ik}^{(n)} - P_{jk}^{(n)} \right| = 0$ $(*)$.

Proof: "coupling".

Want to compare $(X_n^{(1)})_{n\geq 1}$ starting from i

and $(X_n^{(2)})_{n\geq 1}$ starting from j.

Construct a new MC in $S\times S = \bar{S}$

$(X_n^{(1)}, X_n^{(2)})_{n\geq 1}$, where $X_n^{(1)}$ and $X_n^{(2)}$

evolve independently.

$\left(\bar{P}_{(i,j),(k,l)} = P_{ik} \cdot P_{jl} \right)$.

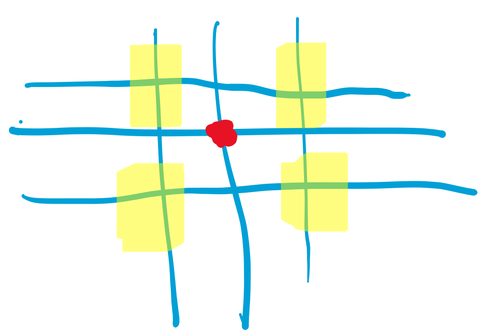
- The joint chain has a stationary distribution

$$\bar{\pi}_{(i,j)} = \pi_i \cdot \pi_j \text{ is stationary}$$

- Irreducible? (Irreducible base chain not enough)

e.g. If P is 1-D SRW

get rid of this by aperiodicity.



Lemma. If i is aperiodic, and $f_{ii} > 0$ then $\exists\, n_0(i) \in \mathbb{N}$, s.t. $p_{ii}^{(n)} > 0 \ (\forall n \geq n_0)$.

$A = \{ n : p_{ii}^{(n)} > 0 \}$ is "simple"

A satisfies "additivity"

$m \in A$, $n \in A$ $\quad p_{ii}^{(m+n)} \geq p_{ii}^{(m)} \cdot p_{ii}^{(n)} > 0$

$\Rightarrow m+n \in A$.

then the result follows Bézout identity in number theory.

Corollary. If irreducible & aperiodic then $\forall i,j \in S$, $\exists\, n_0(i,j) \in \mathbb{N}$,

s.t. $p_{ij}^{(n)} > 0 \ (\forall n > n_0(i,j))$.

$\forall i, j, k, l \in S$

$$p_{(i,j),(k,l)}^{(n)} \overset{?}{>} 0$$

For $n > n_0(i,k)$, $P_{ik}^{(n)} > 0$

$n > n_0(j,l)$ $\quad P_{jl}^{(n)} > 0$

So $n > \max(\cdot,\cdot)$, $\quad P_{(i,j),(k,l)}^{(n)} = P_{ik}^{(n)} \cdot P_{jl}^{(n)} > 0.$

Joint chain has stationary distribution

(irreducible)

$\Downarrow$

Joint chain is recurrent.

Fix any $i_0 \in S$

$$\tau = \inf\{n \geq 0, \; X_n^{(1)} = X_n^{(2)} = i_0\}$$

$$\mathbb{P}_{i,j}(\tau < +\infty) = 1.$$

$$P_{ik}^{(n)} = \mathbb{P}_{i,j}(X_n^{(1)} = k)$$ (under the joint chain)

$$= \sum_{m=1}^{+\infty} \mathbb{P}_{i,j}(X_n^{(1)} = k, \; \tau = m)$$

$$= \boxed{\sum_{m=1}^{n} \mathbb{P}_{ij}\left(X_n^{(1)} = k, \tau = m\right)}$$

$$+ \sum_{m=n+1}^{+\infty} \mathbb{P}_{ij}\left(X_n^{(1)} = k, \tau = m\right).$$

$$\mathbb{P}_{ij}\left(X_n = k, \tau = m\right) = \mathbb{P}_{ij}(\tau = m) \cdot \mathbb{P}_{ij}\left(X_n^{(1)} = k \mid \tau = m\right)$$

$$= \mathbb{P}_{ij}(\tau = m) \cdot P_{i_0 k}^{(n-m)}$$

Exactly the same for $\left(X_n^{(2)}\right)_{n \geq 0}$.

So

$$\left|P_{ik}^{(n)} - P_{jk}^{(n)}\right| \leq \mathbb{P}_{ij}\left(X_n^{(1)} = k, \tau \geq n+1\right)$$

$$+ \mathbb{P}_{ij}\left(X_n^{(2)} = k, \tau \geq n+1\right)$$

$$\leq 2 \mathbb{P}_{ij}(\tau \geq n+1)$$

$$\longrightarrow 0 \qquad (\text{as } n \to +\infty).$$

For convergence rates

- Coupling (delicate construction e.g. card shuffle)
- Eigen-values.

From Markov forgetting to convergence

Idea: compare w/ a chain starting from stationary.

$$\left|p_{ij}^{(n)} - \pi_j\right| = \left|p_{ij}^{(n)} - \sum_{k\in S} \pi_k p_{kj}^{(n)}\right|$$

$$\leq \sum_{k\in S} \pi_k \left|p_{ij}^{(n)} - p_{kj}^{(n)}\right| \to 0$$

M-test

$$\sum_{k\in S} \pi_k \cdot \sup_{n\geq 0} \left|p_{ij}^{(n)} - p_{kj}^{(n)}\right| \leq \sum_{k\in S} \pi_k = 1 < +\infty.$$

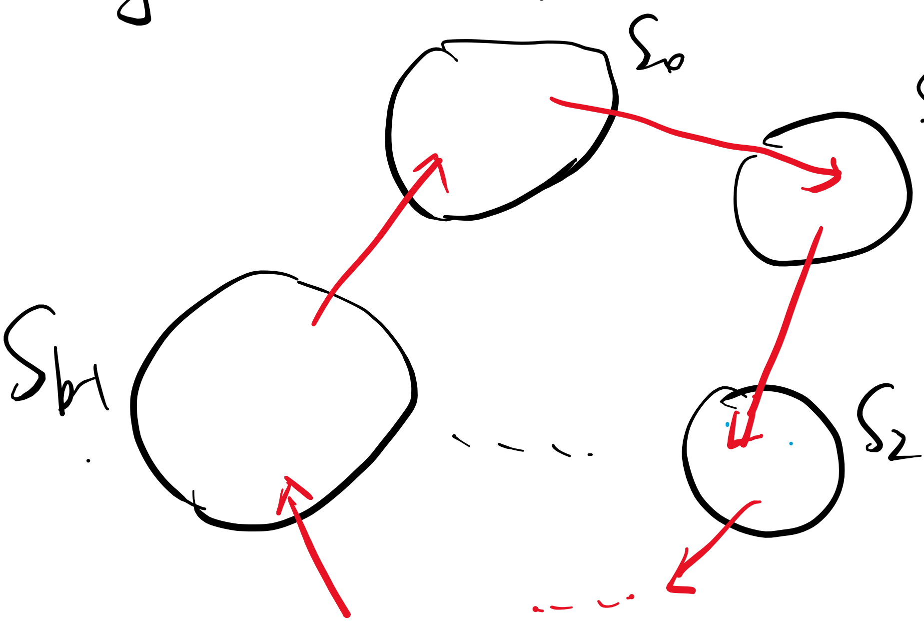
We're allow to interchange $\sum$ w/ lim.

Same arguments apply to MC starting from $X_0 \sim \nu$.

$$\mathbb{P}(X_n = j) \longrightarrow \pi_j$$

How about periodic MC?

"Cycle decomposition". period $b \geq 2$



$$\pi(S_0) = \pi(S_1) = \cdots = \pi(S_{b-1}) = \frac{1}{b}.$$

$$\left(\pi(S) = \sum_{x \in S} \pi(x)\right).$$

Thm. MC is irreducible, period $b \geq 2$, and has stationary distribution π

Then

$$\lim_{n\to+\infty} \frac{1}{b}\left(P_{ij}^{(n)} + P_{ij}^{(n+1)} + \cdots + P_{ij}^{(n+b-1)}\right) = \pi_j.$$

Proof idea:

P^b restricted to each S_i

irreducible, aperiodic, and has stationary.

Corollary, If P is irreducible, and has stationary distribution π,

then

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} P_{ij}^{(n)} = \pi_j$$

Automatically implies: stationary distribution is unique if MC is irreducible. (assuming existence).

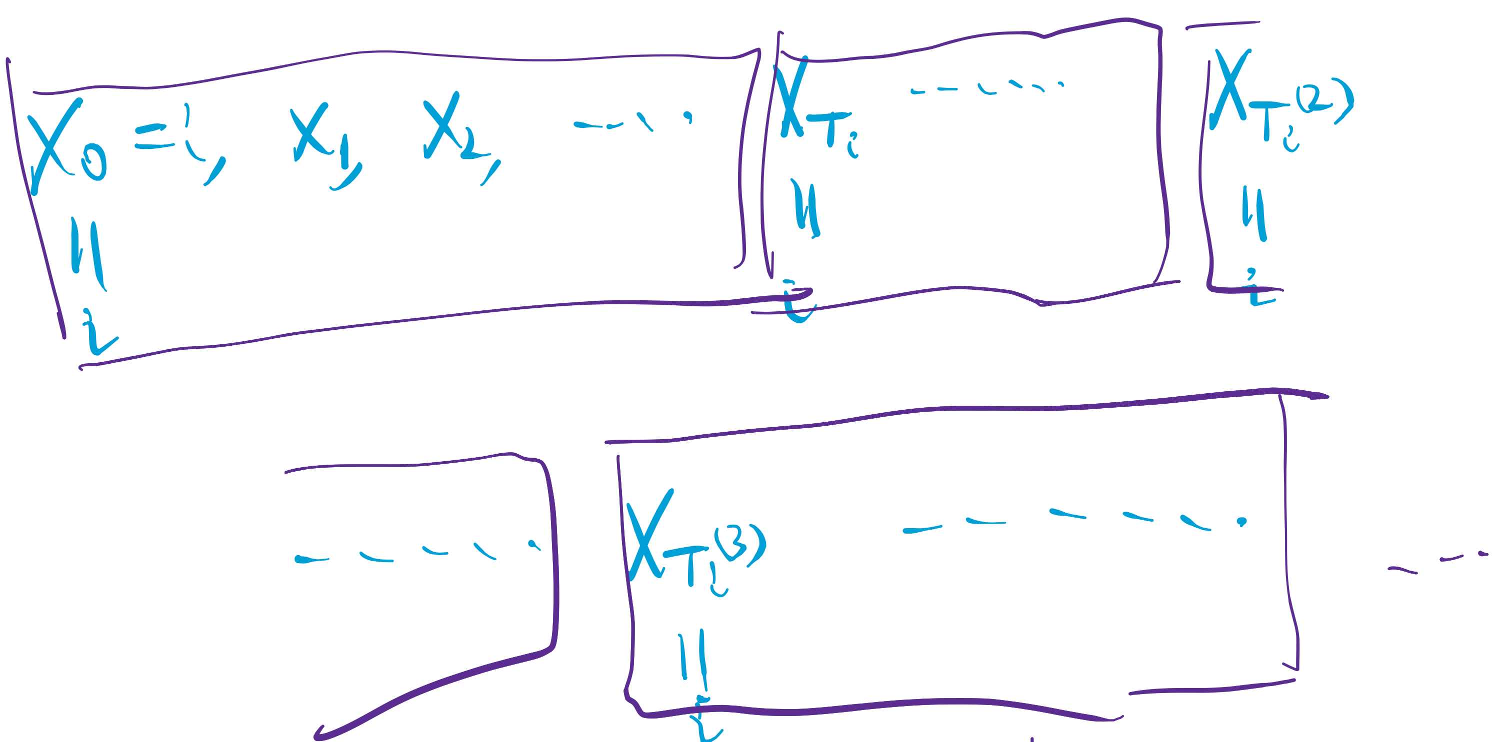
Question:

Clear criteria for existence of stationary distribution / measure?

$$T_i = \inf\{n \geq 1 : X_n = i\}$$

Recurrence $\Longrightarrow$ $\mathbb{P}(T_i < +\infty) = 1.$

$$\mathbb{E}_i[T_i] \overset{?}{<} +\infty.$$



Decomposed to iid blocks

Average length $= \mathbb{E}_i[T_i]$.

Count by

- Blocks : Frequency of i $\to \frac{1}{\mathbb{E}_i[T_i]}$
- Elements : $\to \pi_i$

So we'll have

$$\pi_i = \frac{1}{\mathbb{E}_i[T_i]}$$ if things are "nice".

Thm. If P is irreducible and recurrent, then for any fixed $i_0 \in S$,

$$\forall j \in S, \quad \mu_{i_0}(j) := \sum_{n=0}^{+\infty} \mathbb{P}_{i_0}(X_n = j, T_{i_0} > n).$$

is finite, and μ_{i_0} is a stationary measure of MC. $(\mu_{i_0}(i_0) = 1)$.

i.e. $\mu = \mu P$

Intuition:

$$\mu_{i_0}(j) = \mathbb{E}_{i_0}[\#\text{visits to } j \text{ in } \{0, 1, \cdots, T_{i_0}-1\}].$$

$$\mu_{i_0}P(j) = \mathbb{E}_{i_0}[\#\text{visits to } j \text{ in } \{1, 2, \cdots, T_{i_0}\}]$$

Formal proof.

$$\sum_{j\in S} \mu_{i_0}(j)\cdot P_{jk} = \sum_{n=0}^{+\infty}\sum_{j\in S} \mathbb{P}_{i_0}(X_n = j, T_{i_0} > n)\cdot P_{jk}$$

$$\mathbb{P}_{i_0}(X_n=j, T_{i_0}>n)\cdot \mathbb{P}_j(X_1=k)$$

$$= \begin{cases} \mathbb{P}_{i_0}(X_n=j, X_{n+1}=k, T_{i_0}>n+1) & (k\neq i_0) \\ \mathbb{P}_{i_0}(X_n=j, T_{i_0}=n+1) & (k=i_0) \end{cases}$$

Sum over $j \in S$

$$\begin{cases} \mathbb{P}_{i_0}(X_{n+1}=k, T_{i_0}>n+1) & (k\neq i_0) \\ \mathbb{P}_{i_0}(T_{i_0}=n+1) & (k=i_0) \end{cases}$$

Sum over n

$$\left\{\begin{array}{l}\boxed{\sum_{n=0}^{+\infty} \mathbb{P}_{i_0}\left(X_{n+1}=k,\ T_{i_0}>n+1\right)} \quad (k\neq i_0)\\ \sum_{n=0}^{+\infty} \mathbb{P}_{i_0}\left(T_{i_0}=n+1\right)=1=\mu_{i_0}(i_0) \quad (k=i_0)\end{array}\right.$$

Replace $n+1$ with n (Note that $X_0=i_0\neq k$)

This term $=\mu_{i_0}(k)$.

So $\mu_{i_0}=\mu_{i_0}\cdot P$

Still need to check $\mu_{i_0}(j)<+\infty$ $(\forall j\in S)$.

$$1=\mu_{i_0}(i_0)=\sum_{j\in S}\mu_{i_0}(j)\cdot p^{(n)}(j,i_0)$$

(for a fixed $j\in S$)

$$\geq \mu_{i_0}(j)\cdot p^{(n)}(j,i_0).$$

(Notation $p^{(n)}(j,i_0)=P^{(n)}_{j,i_0}$).

$j\to i_0$ So $\exists\, n>0$, s.t. $p^{(n)}(j,i_0)>0$

and we'll have $\mu_{i_0}(j)\leq\dfrac{1}{p^{(n)}(j,i_0)}<+\infty$.

Fact. $\mu_{i_0}(j) > 0$

Proof of fact: hit lemma

(c.f. reducible MC, stationary measure may be supported on a subset)

When does it become a stationary distribution?

$$\sum_{j\in S} \mu_i(j) = \sum_{j\in S} \sum_{n=0}^{+\infty} \mathbb{P}_i(X_n = j, T_i > n).$$

$$= \sum_{n=0}^{+\infty} \sum_{j\in S} \mathbb{P}_i(X_n = j, T_i > n)$$

$$= \sum_{n=0}^{+\infty} \mathbb{P}_i(T_i > n)$$

$$= \mathbb{E}_i[T_i].$$

When $\mathbb{E}_i[T_i] < +\infty$

$$\pi_j = \frac{\mu_i(j)}{\mathbb{E}_i[T_i]} \quad (\forall j \in S)$$

is a stationary distribution.

Def. A state i is called positive recurrent if $\mathbb{E}_i[T_i] < +\infty$.

Def. A state i is called null recurrent if recurrent but not positive recurrent.

Fact. If $i \leftrightarrow j$, i is positive recurrent then j is also positive recurrent.

Corollary. If P is irreducible, and $i \in S$ is positive recurrent, then all states are pos. rec, $\exists$ a stationary distribution.

When $\mathbb{E}_i[T_i] = +\infty$, $\forall i \in S$, then a stationary distribution does not exist.

Irreducible MC
- transient
- null recurrent $\Leftrightarrow$ $\nexists$ stationary distribution (stat. measure exists)
- positive rec $\Leftrightarrow$ $\exists$ stationary distribution.

We have shown

$$\mathbb{E}_i[T_i] < +\infty \Rightarrow \text{stat. distr.}$$

The other way around?

(**)

Thm. P is irreducible & recurrent,

$$N_n(i) := \sum_{t=1}^{n} \mathbb{1}\{X_t = i\}$$ r.v.

then we have $\frac{N_n(i)}{n} \to \frac{1}{\mathbb{E}_i[T_i]}$ (a.s.).

Remark

- SLLN version of the MC convergence thm.

  c.f. MC convergence (average version)

  $$\boxed{\frac{1}{n}\sum_{t=1}^{n} P_{ij}^{(t)}} \to \pi_j$$

  Deterministic.

- Corollary. P is irreducible and has stat. distr. π then

  $$\pi_i = \frac{1}{\mathbb{E}_i[T_i]} \quad (\forall i \in S)$$

(Completing the proof for "the other way around")

Proof of Corollary from the general theorem.

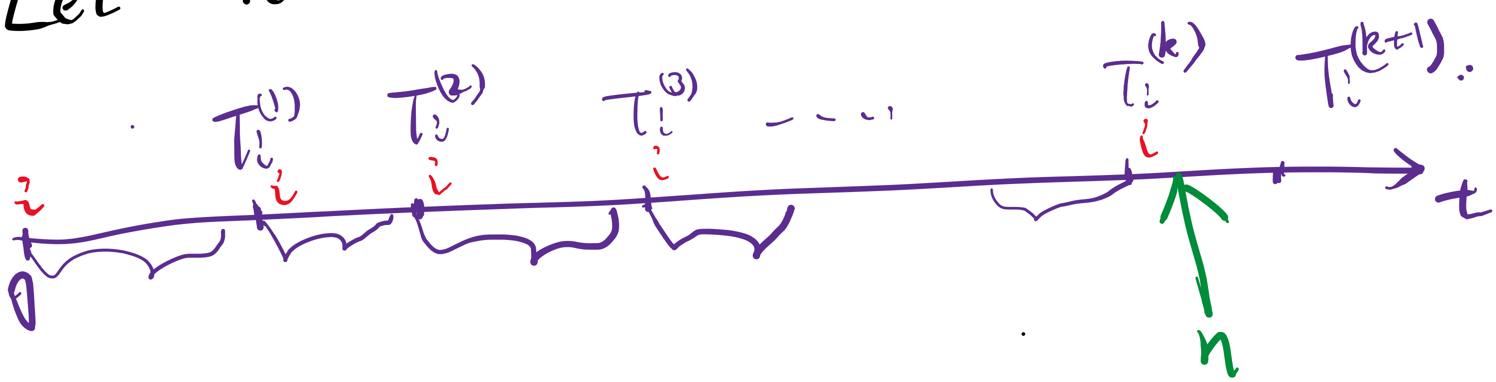
$$\frac{1}{\mathbb{E}_i[T_i]} \leftarrow \frac{\mathbb{E}_i[N_n(i)]}{n} = \frac{1}{n}\sum_{t=1}^{n} P_{ii}^{(t)} \longrightarrow \pi_i$$

Due to (**) and DCT.

by MC convergence thm.

Proof of (**):

Let $T_i^{(k)}$:= time for k-th visit to i.



For $k = 0, 1, 2, \dots$,

$\{X_t : T_i^{(k)} < t \leq T_i^{(k+1)}\}$ are iid blocks.

(Notation: $T_i^{(0)} = 0$).

$k = N_n(i)$

Note that:

$$\frac{T_i^{(N_n(i))}}{N_n(i)} \le \frac{n}{N_n(i)} \le \frac{T_i^{(N_n(i)+1)}}{N_n(i)}$$

By recurrence $N_n(i) \to +\infty$ (a.s.)

as $n \to +\infty$

Boils down to

$$\frac{T_i^{(k)}}{k} \quad (\text{as } k \to +\infty).$$

$$\frac{1}{k} T_i^{(k)} = \frac{1}{k} \sum_{l=1}^{k} \left(T_i^{(l)} - T_i^{(l-1)} \right)$$

$$\xrightarrow{SLLN} \mathbb{E}_i[T_i] \quad (\text{a.s.}).$$

$\begin{cases} \text{If } \mathbb{E}_i[T_i] < +\infty, \text{ converges} \\ \text{If } \mathbb{E}[T_i] = +\infty, \text{ diverges.} \end{cases}$

$$\frac{T_i^{(k)}}{k} \longrightarrow \frac{1}{\mathbb{E}_i[T_i]}$$

$$\frac{T_i^{(k+1)}}{k} = \frac{T_i^{(k+1)}}{k+1} \cdot \frac{k+1}{k}$$

$$\longrightarrow \frac{1}{\mathbb{E}_i[T_i]} \quad \text{a.s.}$$

$\frac{n}{N_n(i)}$ sandwiched

$\Longrightarrow$ also converges (a.s.)