STA 447/2006 Lecture 6

Intuition: "fair gambling".

X_n : money in n-th round

Def. A martingale is a real-valued stochastic process satisfying

$$\mathbb{E}[X_{n+1} \mid X_1, X_2, \cdots, X_n] = X_n$$

(conditional distribution can depend on the entire history).

and $\mathbb{E}[|X_n|] < +\infty$ for any $n = 1, 2, \cdots$

e.g. SRW on $\mathbb{Z}$

$$\mathbb{E}[X_{n+1} \mid X_n] = X_n$$

In general, suppose $Y_1, Y_2, \cdots, Y_n \overset{iid}{\sim} \mathbb{P}$

$\mathbb{E}[|Y_i|] < +\infty$. $\mathbb{E}[Y_i] = 0$

$X_n = \sum_{i=1}^{n} Y_i$ is MG.

(Even if we can verify cond. exp. still need to verify $\mathbb{E}[|X_n|] < +\infty$ to be a MG).

e.g. Let X_n be SRW on $\mathbb{Z}$.

$Y_n = X_n^2$ $\qquad X_n = \sum_{t=1}^{n} Z_t$

$$\mathbb{E}[Y_n \mid X_1, X_2, \cdots, X_{n-1}]$$

$$= \mathbb{E}[(X_{n-1} + Z_n)^2 \mid X_1, \cdots, X_{n-1}]$$

$$= \mathbb{E}[X_{n-1}^2 \mid X_1 \cdots X_{n-1}] + \mathbb{E}[Z_n^2 \mid \cdots] + \boxed{2\mathbb{E}[X_{n-1} Z_n \mid \cdots]}$$

$$= X_{n-1}^2 + 1 + 0$$

$$= Y_{n-1} + 1.$$

Integrability:

$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_n] = n.$

$\mathbb{E}[|Y_n - n|] \leq \mathbb{E}[|Y_n|] + n \leq 2n$

, $(Y_n - n)_{n=0,1,2,\cdots}$ is a MG.

Cross term:

$$\mathbb{E}[X_{n-1} Z_n \mid X_1, X_2 \cdots X_{n-1}]$$

$$= X_{n-1}\, \mathbb{E}[Z_n \mid X_1, X_2, \cdots, X_{n-1}]$$

$$= X_{n-1}\, \mathbb{E}[Z_n]$$

$$= 0.$$

Fact. For $0 \leq m < n$

$$\mathbb{E}[X_n \mid X_0, X_1, X_2 \cdots X_m] = X_m.$$

Proof: eg. $\mathbb{E}[X_{n+2} \mid X_1, \cdots X_n]$

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$. $\Rightarrow$

$$= \mathbb{E}\Big[\underbrace{\mathbb{E}[X_{n+2} \mid X_1, \cdots X_{n+1}]}_{\text{By def of MG.}} \Big| X_1, \cdots X_n\Big]$$

$$= \mathbb{E}[X_{n+1} \mid X_1, \cdots X_n] = X_n$$

In general, prof by induction.

Additional notation:

$\sigma(X_1, X_2, \cdots, X_n)$: "Information contained in $X_1 \cdots X_n$."

$\mathcal{F}_n = \sigma(X_1, \cdots X_n)$. "filtration".

so that a MG satisfies $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = X_n$.

Stopping time:

non-example. $(X_n)_{n \geq 0}$ SRW

Fix $N > 0$, let

$$\tau = \arg\max_{0 \leq t \leq N} \{X_t\}.$$

Key distinction: know/don't know the time is reached when it is reached.

Def. A non-negative-integer-valued r.v. T a stopping time if the event $\{T = n\}$ is determined by $X_0, X_1, \cdots, X_n$ for any $n = 0, 1, 2, \cdots$.

(measurable in $\mathcal{F}_n$)

Can also extend to cts time MG's:

Event $\{T \leq t\}$ is determined by $(X_s)_{0 \leq s \leq t}$ for any $t \geq 0$.

Examples

— For any deterministic $c \in \mathbb{Z}$, $c \geq 0$

c is a stopping time.

(end the game in 10-th round)

— Hitting time :

e.g. $T := \inf\{t \geq 0 : X_t = 5\}$

e.g. $T := \inf\{t \geq 0 : |X_t| \geq 5\}$

e.g. $T := T_i^{(k)}$ k-th visit time to i.

— $T = \inf\{t \geq 0 : X_{t-2} = 5\}$

Non-example :

$$\boxed{T = \inf\{t \geq 0 : X_{t+1} = 5\}}$$

not a stopping time.

Operations. Suppose T_1, T_2 are both stopping times.

- $\min(T_1, T_2)$ ✓
- $\max(T_1, T_2)$ ✓
- $T_1 + T_2$ ✓
- $T_1 \times T_2$ ✓ (only for discret time)

(Suppose $T_1 \geq T_2$)

- $T_1 - T_2$ ✗.

$\mathbb{E}[X_T]$?

$\overset{?}{\|}$

$\mathbb{E}[X_0]$.

- $(X_t)_{t\geq 0}$ MG
- T stopping time.

Counter-example. $(X_t)_{t\geq 0}$ SRW

$T := \inf\{t \geq 0 : X_t = 5\}$. $\mathbb{P}(T < +\infty) = 1$ (by recurrence)

$\mathbb{E}[X_T] = 5 \neq 0$.

Need structures to rule out this case:

e.g. by null recurrence, $\mathbb{E}[T] = +\infty$.

So maybe impose tail assumption on T.

"Easy case": $\mathbb{P}(T \leq m) = 1$ $(*)$

for some deterministic constant $m \geq 0$.

Lemma. If $(X_n)_{n\geq 0}$ is a MG, T stopping time satisfying Eq $(*)$.

then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

(Optional stopping lemma — bounded case).

Proof of Lemma:

$$\mathbb{E}[X_T] - \mathbb{E}[X_0] = \mathbb{E}[X_T - X_0]$$
$$= \mathbb{E}\left[\sum_{k=1}^{T} (X_k - X_{k-1})\right]$$
$$= \mathbb{E}\left[\sum_{k=1}^{m} (X_k - X_{k-1}) \cdot \mathbb{1}_{k \le T}\right]$$
$$= \sum_{k=1}^{m} \mathbb{E}\left[(X_k - X_{k-1}) \mathbb{1}_{k \le T}\right]$$

(Finite summation, always allowed to interchange)

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k-th term $= \mathbb{E}\left[(X_k - X_{k-1}) \mathbb{1}_{k \le T}\right]$

involves info in k-th round?

Actually, $\mathbb{1}_{k \le T}$ only involves info up to $(k-1)$-th round.

$$\mathbb{1}_{k \le T} = 1 - \mathbb{1}_{k \ge T+1}$$
$$= 1 - \sum_{j=0}^{k-1} \mathbb{1}_{\{T=j\}}$$

determined solely by $X_0, X_1, \cdots, X_j$

Put them together, determined by $X_0, X_1, \cdots, X_{k-1}$.

$$\mathbb{E}\left[(X_k - X_{k-1}) \mathbb{1}_{k \le T}\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[(X_k - X_{k-1}) \mathbb{1}_{k \le T} \mid \mathcal{F}_{k-1}\right]\right]$$
$$= \mathbb{E}\left[\mathbb{1}_{k \le T}\, \mathbb{E}\left[X_k - X_{k-1} \mid \mathcal{F}_{k-1}\right]\right]$$
$$= 0.$$

Substituting back completes the proof.

Remark: If we replace m w/ $+\infty$ in the proof, get OST under condition $\sum_{k=1}^{+\infty} \mathbb{E}[|X_k - X_{k-1}| \cdot 1_{k\le T}] < +\infty$

But we can prove OST under weaker conditions.

Thm (Optional Stopping). T is stopping time, $\mathbb{P}(T<+\infty)=1$, $(X_n)_{n\ge 0}$ is MG,

(i) $\mathbb{E}[|X_T|] < +\infty$

(ii) $\lim_{n\to+\infty} \mathbb{E}[|X_n| \cdot 1_{T>n}] = 0$

then OST holds, i.e., $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

(e.g. special case, when T is bounded, $\mathbb{E}[|X_n| \cdot 1_{T>n}] = 0$ for n larger than range of T).

e.g. a useful special case.

Note that w/o $|X_n|$, $\mathbb{E}[1_{T>n}] = \mathbb{P}(T>n) \to 0$ since $\mathbb{P}(T<+\infty)=1$.

Suppose if $|X_n| \le c$ when $n \le T$.

$$\mathbb{E}[|X_n| 1_{T>n}] \le c \cdot \mathbb{P}(T>n) \to 0.$$

this leads to corollary.

Corollary. If $|X_n|$ is bounded by c up to time T, $\mathbb{P}(T<+\infty)=1$. then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

eg. Gambler's ruin revisited.

symmetric case: $X_{n+1} = X_n + \varepsilon_{n+1}$ $\quad \varepsilon_n \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$

$X_0 = a \in (0, c).$

$(X_n)_{n\geq 0}$ MG

$$T := \inf\{t \geq 0 : X_t = 0 \text{ or } X_t = c\}$$

For $n \leq T$ $\quad |X_n| \leq c < +\infty$

So OST holds

$$\mathbb{E}[X_T] = \mathbb{E}[X_0] = a$$

$$\mathbb{E}[X_T] = 0 \cdot \mathbb{P}(X_T = 0) + c \cdot \mathbb{P}(X_T = c)$$

So $\mathbb{P}(X_T = c) = \dfrac{a}{c}.$

- Asymmetric case: $\mathbb{P}(\varepsilon_n = 1) = p, \ \mathbb{P}(\varepsilon_n = -1) = 1-p$ $(p \neq 1/2).$

Construct a martingale:

$$Y_n = \left(\frac{1-p}{p}\right)^{X_n} \quad \text{for } n = 0, 1, \ldots$$

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \left(\frac{1-p}{p}\right)^{X_n} \cdot \left\{ p \cdot \frac{(1-p)}{p} + (1-p) \cdot \frac{p}{1-p} \right\}$$

$$= \left(\frac{1-p}{p}\right)^{X_n} = Y_n$$

Up to time T, (when $n \leq T$).

$$|Y_n| \leq \max\left(1, \left(\frac{1-p}{p}\right)^c\right) < +\infty.$$

So by OST.

$$\left(\frac{1-p}{p}\right)^a = \mathbb{E}[Y_0] = \mathbb{E}[Y_T]$$
$$= \left(\frac{1-p}{p}\right)^c \cdot \mathbb{P}(X_T = c) + \left(\frac{1-p}{p}\right)^0 \cdot \mathbb{P}(X_T = 0)$$

Solve for $\mathbb{P}(X_T = c)$.

Back to the main theorem.

Proof. Idea: truncation and use lemma.

$$T_m = \min(T, m) \quad \text{for any } m = 0, 1, \ldots$$

$0 \leq T_m \leq m$ is a stopping time.

We know from lemma: $\mathbb{E}[X_{T_m}] = \mathbb{E}[X_0]$.

$$X_{T_m} = \boxed{X_T \cdot \mathbb{1}_{\{T \leq m\}}} + X_m \cdot \mathbb{1}_{\{T > m\}}$$
$$X_T = \boxed{X_T \cdot \mathbb{1}_{\{T \leq m\}}} + X_T \cdot \mathbb{1}_{\{T > m\}}.$$

Cancelled when taking difference.

$$\text{Error} = \left|\mathbb{E}[X_{T_m}] - \mathbb{E}[X_T]\right|$$
$$= \left|\mathbb{E}[X_m \mathbb{1}_{\{T > m\}}] - \mathbb{E}[X_T \cdot \mathbb{1}_{T > m}]\right|$$
$$\leq \mathbb{E}[|X_m| \mathbb{1}_{T > m}] + \mathbb{E}[|X_T| \cdot \mathbb{1}_{T > m}]$$

Want to show Error $\to 0$ as $m \to +\infty$.

- $\mathbb{E}[|X_m| 1_{T>m}] \to 0$ as assumed. (ii)
  Both parts are varying w/ m.
- $\mathbb{E}[|X_T| 1_{T>m}]$ only the indicator depends on m.

$$|X_T| \cdot 1_{T>m} \le |X_T|. \qquad \mathbb{E}[|X_T|] < +\infty$$
by assumption (i).

Since $\mathbb{P}(T<+\infty)=1$, $\mathbb{P}(1_{T>m} \to 0 \text{ as } m \to +\infty) = 1$.

By DCT, $\mathbb{E}[|X_T| 1_{T>m}] \to 0$ (as $n \to +\infty$).

More examples.

Gambler's ruin problem.

$$\mathbb{E}[T] = ?$$

- Symmetric case. $(X_n)_{n\ge 0}$ is MG
  but applying OST does not give info about $\mathbb{E}[T]$.

Idea: construct another MG.

Let $S_n = X_n^2 - n$.

$(S_n)_{n\ge 0}$ is a MG (we already proved this)

If we can apply OST

$$a^2 = \mathbb{E}[S_0] = \mathbb{E}[S_T]$$
$$= \mathbb{E}[X_T^2] - \mathbb{E}[T]$$
$$= \underbrace{c^2 \mathbb{P}(X_T = c)}_{c^2 \cdot \frac{a}{c}} + \cancel{0^2 \cdot \mathbb{P}(X_T = 0)} - \mathbb{E}[T]$$

$$\mathbb{E}[T] = a \cdot (c-a).$$

(more generally, study MGF of T using exponential martingales).

It remains to verify the assumptions.

$S_n = X_n^2 - n$ is not unif bdd up to time T.

$$\lim_{n \to +\infty} \mathbb{E}[|S_n| \cdot 1_{T>n}] \overset{?}{\to} 0.$$

$$|S_n| \le c^2 + n$$

Need to bound

$$\mathbb{E}[(c^2+n) 1_{T>n}] \le \underbrace{c^2 \cdot \mathbb{P}(T>n)}_{\to 0} + \mathbb{E}[T \cdot 1_{T>n}]$$

(Since $T < +\infty$ a.s.)

$$\mathbb{E}[T 1_{T>n}] \overset{?}{\to} 0.$$

By DCT, it remains to show $\mathbb{E}[T] < +\infty$.

For gambler's ruin problem,

Claim. $\mathbb{P}(T \geq n) \leq C \cdot \rho^n$

for some $C > 0$ and $\rho \in (0,1)$.

(True in general for finite state space MC's).

(see also, Lawler's exercise question).