STA 447/2006 Lecture 7

Recall OST

$(X_t)_{t\geq 0}$ be MG and T is a stopping time.

$$\mathbb{E}[X_T] = \mathbb{E}[X_0]$$

when (i) $\mathbb{E}[|X_T|] < +\infty$

(ii) $\mathbb{E}[|X_n| \cdot \mathbb{1}_{T>n}] \to 0$ (as $n \to +\infty$).

(Useful special cases)

— T bounded

— $\{|X_n| : n \leq T\}$ uniformly bounded by some const.

Application to gambler's ruin (symmetric).

Interested in $\mathbb{E}[T]$,

where T is the terminal time.

From last lecture:

- $M_n = X_n^2 - n$ is an MG.
- If OST holds true, we have

$$\mathbb{E}[T] = a(c-a)$$

($a = X_0$, $c > a$ is the target amount).

Neither of special cases apply.

Work w/ original version.

$$\mathbb{E}\left[|M_n| \cdot \mathbb{1}_{T>n}\right] \le (C^2+n) \cdot \mathbb{E}\left[\mathbb{1}_{T>n}\right]$$
$$= (C^2+n) \cdot \mathbb{P}(T>n).$$

$\to 0$ since T is finite (w.p. 1).

Need it to converge to 0 at certain rate (faster than $1/n$).

Using MC theory, we can verify

$$\mathbb{P}(T>n) \le C \cdot \rho^n \quad \text{for some } C>0, \ \rho \in (0,1)$$

(holds true for any finite statespace MC).

Proof Idea: starting from any $i \in S$, $\exists$ $n_i > 0$

s.t. $q_i := \mathbb{P}_i(\text{hitting target in } n_i \text{ steps}) > 0.$

Key observation: finite $|S|$,

take $n := \max_i n_i$ and $q := \min_i q_i$.

For any i,

$$\mathbb{P}_i(\text{hitting target in } n \text{ steps}) \ge q.$$

$\mathbb{P}($succeed in n steps$) \ge q$

"hitting target"

$\mathbb{P}(\text{succeed in } [n+1, 2n] \mid \text{unsuccessful in } [0,n]) \geq q.$

$\mathbb{P}(\text{unsuccessful in first } m \cdot n \text{ steps}).$

$= \mathbb{P}(\text{unsuccessful in } [(m-1)n+1, mn] \mid \text{unsuccessful in } [(m-2)n+1, (m-1)n])$

$\cdot \mathbb{P}(\text{-----} [(m-2)n+1, (m-1)n] \mid \text{-----} [(m-3)n+1, (m-2)n])$

......

$\leq (1-q)^m.$

$(n \text{ is fixed})$

$\mathbb{P}(T > N) \leq (1-q)^{N/n}$ So $C = 1$

So $\mathbb{P}(T > mn) \leq (1-q)^m.$

$\rho = (1-q)^{1/n}.$

Using this fact, we can verify

$$(n+C^2) \cdot \mathbb{P}(T > n) \leq (n+C^2) \cdot \rho^n \longrightarrow 0$$

(since $\rho < 1$). So OST applies.

A closer look at the limit condition.

$$\mathbb{E}[\underbrace{|X_n|}_{\forall n, \text{ integrable r.v.}} \underbrace{\mathbb{1}_{\{T>n\}}}_{\mathbb{P}(T>n) \to 0.}]$$

Recall in prob theory. (DCT).

$$\mathbb{E}[\,|X|\,\mathbb{1}_{A_n}] \to 0$$

as $n \to +\infty$ when $\mathbb{P}(A_n) \to 0$
and $\mathbb{E}[|X|] < +\infty$.

Idea: impose additional assumptions on the sequence $(X_n)_{n\geq 0}$ s.t. they behave like a single r.v. in the tail.

Def. A sequence of integrable r.v.'s $(X_n)_{n\geq 0}$ is called uniformly integrable (u.i.) if $\forall \varepsilon > 0$, $\exists K > 0$, s.t. $\mathbb{E}[\,|X_n|\,\mathbb{1}_{|X_n|\geq K}] \leq \varepsilon$ $(\forall n)$.

(Note: K cannot depend on n).

Thm. If $(X_n)_{n\geq 0}$ u.i., stopping time T satisfies $\begin{cases} T < +\infty \ (a.s.) \\ \mathbb{E}[|X_T|] < +\infty. \end{cases}$

then we have $\mathbb{E}[\,|X_n|\,\mathbb{1}_{T>n}] \to 0$

and therefore, $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Remark: only need u.i. MG $(X_{n\wedge T})_{n\geq 0}$

(Notation $x \wedge y := \min(x, y)$)

Proof: Define the event $A_n := \{T > n\}$.

$$\mathbb{E}[\,|X_n|\cdot\mathbb{1}_{A_n}] = \mathbb{E}[\,|X_n|\cdot \mathbb{1}_{A_n\cap\{|X_n|>K\}}] + \mathbb{E}[\,|X_n|\cdot\mathbb{1}_{A_n\cap\{|X_n|\leq K\}}]$$

$$\leq \underbrace{\mathbb{E}[\,|X_n|\cdot\mathbb{1}_{|X_n|>K}]}_{\leq \varepsilon \text{ by u.i.}} + K\cdot\mathbb{P}(A_n)$$

Let $K := K_\varepsilon$ be the K in u.i. condition.

$$\mathbb{E}[|X_n| \cdot \mathbb{1}_{A_n}] \leq \varepsilon + K \cdot \mathbb{P}(A_n)$$

Take $n \to +\infty$,

$$\lim_{n\to+\infty} \mathbb{E}[|X_n| \mathbb{1}_{A_n}] \leq \varepsilon \quad (\forall \varepsilon > 0)$$

Since ε is arbitrary, $\lim_{n\to+\infty} \mathbb{E}[|X_n| \mathbb{1}_{A_n}] = 0.$

Corollary: If $\exists C < +\infty$ s.t. $\mathbb{E}[|X_n|^2] \leq C$ for any n,

then $(X_n)_{n \geq 0}$ is u.i.

(Remark: first moment bound is NOT enough)

Proof:

$$\mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geq K}]$$

$$\overset{(C\text{-}S)}{\leq} \sqrt{\mathbb{E}[X_n^2]} \cdot \sqrt{\mathbb{E}[\mathbb{1}^2_{|X_n| \geq K}]}$$

$$= \sqrt{\mathbb{E}[X_n^2]} \cdot \sqrt{\mathbb{P}(|X_n| \geq K)}$$

$$\leq \sqrt{C \cdot \mathbb{P}(|X_n|^2 \geq K^2)}$$

(Markov's ineq)

$$\leq \sqrt{C \cdot \frac{C}{K^2}}$$

$$= C/K$$

$\forall \varepsilon > 0$, we take $K := C/\varepsilon$, u.i.

e.g. $X_n = \sum_{j=1}^{n} \frac{1}{j} Z_j$ where $Z_j \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$

$$\mathbb{E}[|X_n|^2] = \sum_{j=1}^{n} \frac{1}{j^2} \le \sum_{j=1}^{+\infty} \frac{1}{j^2} = \frac{\pi^2}{6} < +\infty.$$

So $(X_n)_{n\ge 1}$ is u.i.

e.g. (Non-example). X_n 1-D symmetric SRW.

$T :=$ hitting time of -1.

$Y_n = 1 + X_{n\wedge T}$ for $n = 0, 1, 2, \ldots$

$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_n] = \mathbb{E}[Y_0] = 1 < +\infty.$

But $(Y_n)_{n\ge 0}$ is not u.i. (will see later).

Back to OST.

Wald's identity (OST but w/ special structures, play w/ truncation arguments manually).

$X_n = \sum_{j=1}^{n} Z_j$ where Z_j's are iid, $\mathbb{E}[|Z_j|] < +\infty$, $\mathbb{E}[Z_j] = m$.

$(X_n - nm)_{n\ge 0}$ is an MG.

Thm (Wald). If T is a stopping time satisfying $\mathbb{E}[T] < +\infty$, then, $\mathbb{E}[X_T] = m\,\mathbb{E}[T]$.

Proof. By "optional stopping lemma" Applied to stopping time $T_n := n \wedge T$.

$$\forall n, \quad \mathbb{E}[X_{n\wedge T}] = m \cdot \mathbb{E}[n \wedge T].$$

$$0 = \mathbb{E}[X_{n\wedge T} - m(n\wedge T)]$$

$$\lim_{n\to+\infty} \mathbb{E}[n\wedge T] = \mathbb{E}[T]$$ since T is integrable.

Study the truncation error.

$$\left| \mathbb{E}[X_{n\wedge T}] - \mathbb{E}[X_T] \right|$$

$$\leq \mathbb{E}[|X_{n\wedge T} - X_T|]$$

$$= \mathbb{E}\left[\left| \mathbb{1}_{\{T>n\}} \cdot \sum_{m=n+1}^{T} Z_m \right|\right]$$

$$\leq \mathbb{E}\left[\sum_{m=n+1}^{T} |Z_m| \cdot \mathbb{1}_{T\geq m}\right]$$

$$= \mathbb{E}\left[\sum_{m=n+1}^{+\infty} |Z_m| \cdot \mathbb{1}_{T\geq m}\right]$$

(Fubini-Tonelli)
$$= \sum_{m=n+1}^{+\infty} \mathbb{E}[|Z_m| \cdot \mathbb{1}_{T\geq m}]$$

Key observation:

$$\{T \geq m\} = \{T \leq m-1\}^c = \left(\bigcup_{j=1}^{m-1} \{T = j\}\right)^c$$

is determined by $Z_1, Z_2, \cdots, Z_{m-1}$. and therefore $\{T\geq m\}$ is independent of Z_m.

$$\text{Above} = \sum_{m=n+1}^{+\infty} \mathbb{E}[|Z_m|] \cdot \mathbb{P}(T \geq m).$$

$$= \underbrace{\mathbb{E}[|Z_1|]}_{<+\infty} \cdot \boxed{\sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m)} \to 0\ ?$$

Fact. $\sum_{m=1}^{+\infty} \mathbb{P}(T \geq m) = \mathbb{E}[T] < +\infty.$

So the tail sum $\sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m) \to 0.$

completing proof of Wald's theorem.

Another application: Doob's maximal ineq.

From 347, Markov's ineq. $X \geq 0$

$$\mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}[X]}{\lambda} \quad (\forall \lambda > 0).$$

For MG, $(X_n)_{n \geq 0}$ is non-neg MG,

$$\mathbb{P}(X_n \geq \lambda) \leq \frac{\mathbb{E}[X_0]}{\lambda}.$$

How about

$$\mathbb{P}\left(\max_{1 \leq t \leq n} X_t \geq \lambda\right) ?$$

Thm (Doob)

$$\mathbb{P}\left(\max_{1\le t\le n} X_t \geq \lambda\right) \leq \frac{\mathbb{E}[X_0]}{\lambda}.$$

Proof. Let

$$T := \inf\{t > 0 : X_t \geq \lambda\}.$$

Interested in $\mathbb{P}(T \leq n)$.

By optional stopping lemma,

$$\mathbb{E}[X_{T\wedge n}] = \mathbb{E}[X_0]$$

(Bounded stopping time $T \wedge n \leq n$).

On the other hand

non-negative.

$$\mathbb{E}[X_{T\wedge n}] = \cancel{\mathbb{P}(T>n)\cdot\mathbb{E}[X_{T\wedge n} \mid T>n]}$$
$$+ \mathbb{P}(T\leq n)\cdot\mathbb{E}[X_{T\wedge n} \mid T \leq n]$$

($X_{T\wedge n} \geq \lambda$ when $T \leq n$ happens)

$$\geq \mathbb{P}(T\leq n)\cdot\lambda.$$

So $\mathbb{P}(T \leq n) \leq \dfrac{\mathbb{E}[X_0]}{\lambda}$.

Martingale convergence.

Note: different from MC convergence.

MC convergence, $\lim_{n \to +\infty} P_{ij}^{(n)} = \pi_j$

MG convergence $\mathbb{P}(X_n \to X_\infty) = 1$.

For MG, interested in a.s. convergence.

e.g. Gambler's ruin,

$(X_{n \wedge T})_{n \geq 0}$ is an MG

(T is the hitting time of $\{0, c\}$)

$$X_\infty = \begin{cases} c & \text{w.p. } a/c \\ 0 & \text{w.p. } 1 - a/c \end{cases}$$

e.g. $\sum_{j=1}^{+\infty} Z_j / j$ $\quad Z_j \overset{\text{iid}}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$

w/o the sign — divergence.

But the randomness makes it converge.

Thm. Let $(X_n)_{n\geq 0}$ be an MG.

If $\mathbb{E}[|X_n|] \leq C < +\infty \quad (\forall n)$

then there exists an r.v. X_∞

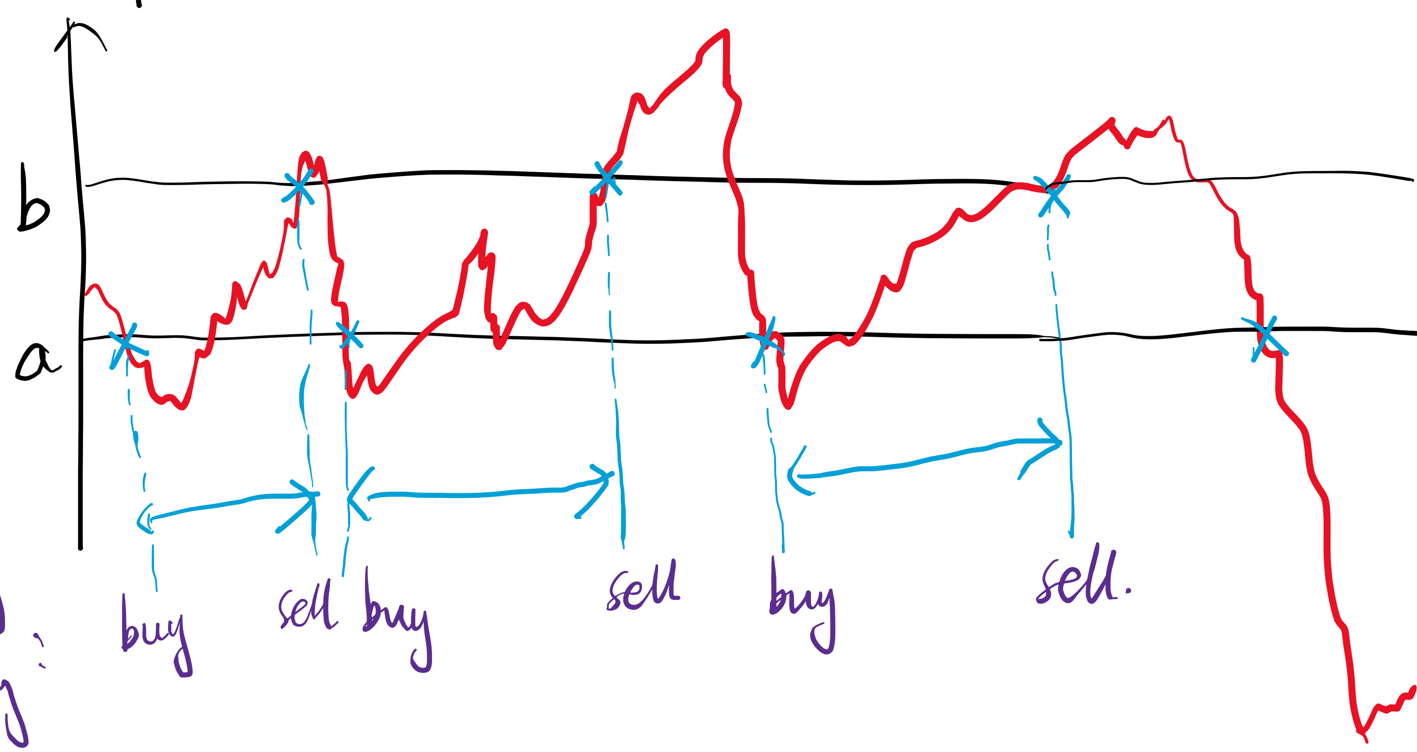
s.t. $\mathbb{P}(X_n \to X_\infty) = 1$.

(Strictly weaker condition than u.i.)

Proof idea: "up-crossing".

Fix $a<b$

Trading strategy:



almost sure convergence

$\Updownarrow$

$\mathbb{P}\left(\forall a, b \in \mathbb{R}, a < b, \# \text{ upcrossings of } [a,b] \text{ is finite}\right) = 1$

Bounding up-crossings.

Use the trading strategy.

$W_n :=$ amount of money made using this strategy

$$= \sum_{j=1}^{n} B_j (X_j - X_{j-1}).$$

where $B_j = \begin{cases} 1 & \text{when } X_{j-1} \le a,\ B_{j-1} = 0 \quad \text{“Buy”} \\ 0 & \text{when } X_{j-1} \ge b,\ B_{j-1} = 1 \quad \text{“Sell”} \\ B_{j-1} & \text{otherwise} \quad \text{“Hold”}. \end{cases}$

Fact: $(W_n)_{n\geq 0}$ is an MG.

$$\mathbb{E}[W_{n+1} | \mathcal{F}_n] = W_n + \mathbb{E}[B_{n+1}(X_{n+1} - X_n) | \mathcal{F}_n].$$

B_{n+1} is determined by history.

So $\mathbb{E}[B_{n+1}(X_{n+1} - X_n) | \mathcal{F}_n]$

$$= B_{n+1} \cdot \mathbb{E}[(X_{n+1} - X_n) | \mathcal{F}_n] = 0.$$

Let U_n be # up-crossings of $[a,b]$ up to time n.

$$W_n \geq (b-a) U_n - |X_n - a|$$

$$\mathbb{E}[W_n] \geq (b-a)\mathbb{E}[U_n] - \mathbb{E}[|X_n - a|]$$

$$0 = \mathbb{E}[W_0] \quad (= \mathbb{E}[W_n])$$

So $\forall n$, we have

$$\mathbb{E}[U_n] \leq \frac{1}{b-a}\left(\mathbb{E}[|X_n|] + a\right) \leq \frac{a+C}{b-a}.$$

The upper bound is indp of n.

So # up-crossings for the entire process $< +\infty$ w.p. 1. ($\forall a, b$ pair).

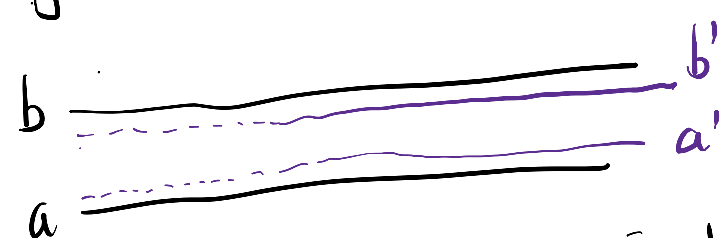
From up-crossing ineq to convergence:

$$\mathbb{P}(\exists a,b \in \mathbb{Q} : \#\text{up-crossing } [a,b] = +\infty)$$

$$\leq \sum_{a,b \in \mathbb{Q}} \mathbb{P}(\#\text{ up-crossing } [a,b] = +\infty) = 0.$$

For real a, b pair.

up-crossing $[a,b]$ will also upcross $[a', b']$.



So $\mathbb{P}(\exists\, a,b \in \mathbb{R},\ \text{up crossing } [a,b] = +\infty) = 0.$