STA447/2006 Lecture 8

Midterm 2: next week.

— Covering Lectures 6, 7, 8

(may involve some MC as e.g. problem background)

(You're allowed to use any technique).

From last time: MG convergence

Thm. $(X_n)_{n\geq 0}$ is MG, and $\mathbb{E}[|X_n|] \leq C < +\infty$ $\forall n$

Then ~~there~~ exists an r.v. X_∞

s.t. $\mathbb{P}(X_n \to X_\infty) = 1$.

Question: $\mathbb{E}[X_\infty] \overset{?}{=} \mathbb{E}[X_n]$

(or in general, $\mathbb{E}[|X_n - X_\infty|] \to 0$).

$|\mathbb{E}[X_\infty] - \mathbb{E}[X_n]| \leq \mathbb{E}[|X_n - X_\infty|]$

So L^1 convergence implies eq. expectation.

Counter-example:

Let $(Z_n)_{n\geq 0}$ be SSRW, and $X_n = Z_n + 1$.

Let $T := \inf\{t \geq 0 : X_t = 0\}$

Let $Y_n := X_{n \wedge T}$ (stopped MG is MG).

$Y_n \geq 0$, MG.

$$\mathbb{E}[|Y_n|] = \mathbb{E}[Y_n] = 1. < +\infty$$

So $Y_n \to Y_\infty$ a.s.

[Indeed, the convergence $Y_n \to 0$ a.s. immediately flows as $T < +\infty$ w.p. 1].

But $1 = \mathbb{E}[Y_0] = \mathbb{E}[Y_\infty] = 0.$

Need additional assumption.

Recall u.i. $\forall \varepsilon > 0$, $\exists K$

s.t. $\mathbb{E}[|X_n| \mathbb{1}_{|X_n| > K}] \leq \varepsilon \quad \forall n \geq 0$

(used eg in OST).

Thm $(X_n)_{n \geq 0}$ u.i., $X_n \xrightarrow{a.s.} X_\infty$ (It is guaranteed that $\mathbb{E}|X_\infty| < +\infty$ by Fatou's lemma).

then we have $\mathbb{E}[|X_n - X_\infty|] \to 0.$

Proof. $\mathbb{E}[|X_n - X_\infty|]$

$$\leq \underbrace{\mathbb{E}[|X_n \mathbb{1}_{|X_n| \leq K} - X_\infty \mathbb{1}_{|X_\infty| \leq K}|]}_{\to 0 \text{ for fixed } K \text{ by DCT}}$$

$$+ \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geq K}] + \mathbb{E}[|X_\infty| \cdot \mathbb{1}_{|X_\infty| \geq K}].$$

Since $(X_n)_{n \geq 0}$ is u.i. and $\mathbb{E}|X_\infty| < +\infty$, $\forall \varepsilon > 0$ fixed, $\exists K = K_\varepsilon > 0$ s.t. $\mathbb{E}[|X_n| \mathbb{1}_{|X_n| \geq K}] \leq \varepsilon \quad \forall n$

$\mathbb{E}[|X_\infty| \mathbb{1}_{|X_\infty| \geq K}] \leq \varepsilon.$

For $\varepsilon > 0$, fix it and find $K = K_\varepsilon < +\infty$

Let $n \to +\infty$, $\limsup_{n \to +\infty} \mathbb{E}\left[|X_n - X_\infty|\right] \leq 2\varepsilon$

Since ε can be arbitrarily small, we have

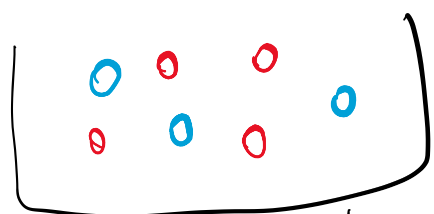
$$\lim_{n \to +\infty} \mathbb{E}\left[|X_n - X_\infty|\right] = 0.$$

eg. $M_n = \sum_{j=1}^{n} \frac{1}{j} X_j$ where $X_j \overset{\text{iid}}{\sim} \begin{cases} 1 & \text{w.p } \frac{1}{2} \\ -1 & \text{w.p } \frac{1}{2}. \end{cases}$

$\mathbb{E}\left[|M_n|^2\right] \leq \frac{\pi^2}{6} < +\infty \quad \forall n$ So u.i.

$$M_n \xrightarrow[L^1]{a.s.} M_\infty \qquad \mathbb{E}[M_\infty] = 0.$$

eg. Poly's Urn.



Let $M_n :=$ Proportion of red balls in the box

For each time, add a new ball

w/ colour $\begin{cases} \text{red} & \text{w.p } M_n \\ \text{blue} & \text{w.p. } 1 - M_n. \end{cases}$

Easy to verify: $(M_n)_{n \geq 0}$ is MG.

uniformly bdd in $[0, 1]$

So $M_n \xrightarrow[L^1]{a.s.} M_\infty$.

Further application. OST and MG convergence.

$(X_n)_{n\geq 1}$ is a u.i MG

T is a stopping time $\mathbb{E}[|X_T|] < +\infty$.

(don't require $\mathbb{P}(T < +\infty) = 1$)

X_∞ indicates the limiting r.v.

$$\mathbb{E}[X_T] \overset{?}{=} \mathbb{E}[X_0]. \quad (*)$$

To show $(*)$, consider $(X_{T\wedge n})_{n\geq 0}$ MG

and we have $X_{T\wedge n} \xrightarrow{a.s.} X_T$.

We can easily verify $(X_{T\wedge n})_{n\geq 0}$ u.i.

$$\mathbb{E}[|X_{T\wedge n}| \mathbb{1}_{|X_{T\wedge n}| \geq K}] \leq \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| \geq K}] + \mathbb{E}[|X_T| \mathbb{1}_{|X_T| \geq K}]$$

So we have $\mathbb{E}[X_T] = \mathbb{E}[X_{T\wedge n}]$ (by L^1 converge)

and therefore $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

How about L^p convergence for $p>1$?

Tool. $$\mathbb{P}\left(\max_{0\leq t\leq n} |X_t| \geq a\right) \leq \frac{\mathbb{E}[|X_n|^p]}{a^p}$$

Doob's L^p max ineq

(Proof similar to L^1 version)

Doob's ineq implies

$$\mathbb{E}\left[\max_{0\le t\le n}|X_t|^p\right] \le \frac{p}{p-1}\cdot\mathbb{E}\left[|X_n|^p\right]$$

for any MG $(X_t)_{t\ge 0}$.

(Proof: integrate w.r.t. a and truncation).

From max ineq to L^p convergence.

$$\mathbb{E}\left[|\underbrace{X_n - X_\infty}_{\to 0 \text{ a.s.}}|^p\right]$$

Need a dominating function.

$$|X_n - X_\infty|^p \le 2^{p-1}\left(|X_n|^p + |X_\infty|^p\right)$$
$$\le 2^{p-1}\left(\sup_{n\ge 0}|X_n|^p + |X_\infty|^p\right)$$

Apply DCT, get L^p convergence.

Conclusion: If $\mathbb{E}\left[|X_n|^p\right] \le C < +\infty$ $(p>1)$

Then $X_n \xrightarrow[L^p]{a.s.} X_\infty$.

Brownian Motion

Motivation: scaling limit of SRW.

$$X_n = \sum_{i=1}^{n} \varepsilon_i \quad \text{where} \quad \varepsilon_i = \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$$

- By CLT. $\frac{1}{\sqrt{n}} X_n \xrightarrow{d} \mathcal{N}(0,1)$.
- For finitely many $0 \le t_1 < t_2 < \cdots < t_k < +\infty$

$$\left(X_{\lfloor nt_1 \rfloor}, X_{\lfloor nt_2 \rfloor}, \cdots, X_{\lfloor nt_k \rfloor} \right).$$

$$\frac{1}{\sqrt{n}} \begin{bmatrix} X_{\lfloor nt_1 \rfloor} \\ X_{\lfloor nt_2 \rfloor} - X_{\lfloor nt_1 \rfloor} \\ \vdots \\ X_{\lfloor nt_k \rfloor} - X_{\lfloor nt_{k-1} \rfloor} \end{bmatrix} \xrightarrow{d} \mathcal{N}\left(0, \begin{bmatrix} t_1 & & 0 \\ & t_2 - t_1 & \\ 0 & & \ddots \\ & & & t_k - t_{k-1} \end{bmatrix}\right).$$

Hope: $\left(\frac{1}{\sqrt{n}} X_{\lfloor nt \rfloor} \right)_{0 \le t \le T} \xrightarrow{d}$ something.

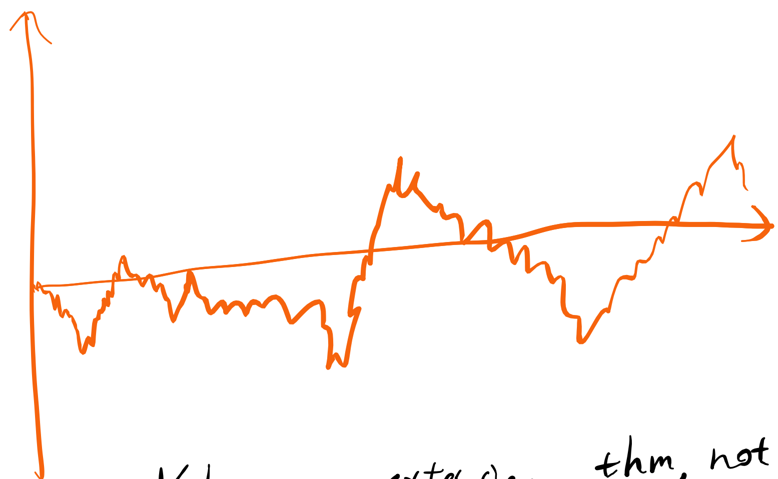
Defn. $(B_t)_{t \ge 0}$ is a Brownian Motion if.

1. $B_0 = 0$
2. For any $t_1, \cdots, t_k$, $(B_{t_1}, B_{t_2}, \cdots, B_{t_k})$ jointly normal.
3. Independent increments.

   For $t > s$, $B_t - B_s \sim \mathcal{N}(0, t-s)$, independent from $(B_r)_{0 \le r \le s}$.

4. $t \mapsto B_t$ is a continuous function w.p. 1.

Intuitively: within infinitesimal time interval $[t, t+\Delta t]$, make an indep inc $\sim \mathcal{N}(0, \Delta t)$.



(Existence: by Kolmogorov extension thm, not covered).

Fact. $(B_t)_{t \geq 0}$ is a martingale.

cts-time martingales: $(X_t)_{t \geq 0}$

- $\mathbb{E}[|X_t|] < +\infty$
- $\mathbb{E}[X_t | \mathcal{F}_s] \left(= \mathbb{E}[X_t | (X_r)_{0 \leq r \leq s}]\right) = X_s$.

Stopping time T: the event $\{T \leq t\}$ determined by $(X_s)_{0 \leq s \leq t}$.

e.g. hitting time of a set.

e.g. $T_1 \wedge T_2$, $T_1 \wedge T_2$, where T_1 and T_2 are stopping times.

OST in continuous time $(X_t)_{t\geq 0}$ is MG

T is a stopping time

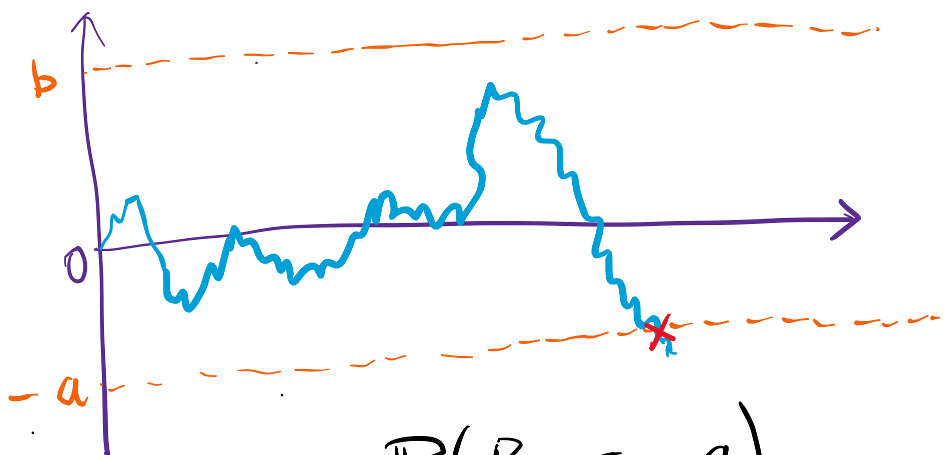
$$\begin{cases} \cdot\ \mathbb{E}[|X_T|] < +\infty \\ \cdot\ \lim_{t\to+\infty} \mathbb{E}\left[|X_t| \cdot \mathbb{1}_{\{T > t\}}\right] = 0 \end{cases}$$

Then $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Application: gambler's ruin in continuous space.

$(B_t : t \geq 0)$ BM. (for $a, b > 0$)

$$T := \inf\{t \geq 0 : B_t = -a \text{ or } B_t = b\}$$



Interested in $\mathbb{P}(B_T = -a)$.

Solution: OST. Note that $\begin{cases} (B_t)_{t\geq 0} \text{ is a MG} \\ |B_t \mathbb{1}_{t\leq T}| \leq \max\{a, b\} \end{cases}$

"Bounded up to time T".

$$0 = \mathbb{E}[B_0] = \mathbb{E}[B_T]$$

$$= -a \cdot \mathbb{P}(B_T = -a) + b \cdot \mathbb{P}(B_T = b)$$

So we get $\mathbb{P}(B_T = -a) = \dfrac{b}{a+b}$.

Remark: requires $\mathbb{P}(T < +\infty) = 1$ ⇝ $\boxed{\mathbb{P}(B_t \in [-a,b]) \to 0}$

Can be shown by — From SRW and MC theory

Starting from any $x \in [-a,b]$,

T_0 large deterministic time

$$\mathbb{P}(\text{escape } [-a,b] \text{ within } T_0) > c$$

$$\Downarrow$$

$$\mathbb{P}(T \geq kT_0) \leq (1-c)^k.$$

How about $\mathbb{E}[T]$?

Define: $M_t := B_t^2 - t$.

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = \mathbb{E}\left[(B_s + (B_t - B_s))^2 \mid \mathcal{F}_s\right] - t$$

$$= B_s^2 + (t-s) - t = B_s^2 - s = M_s$$

$$\mathbb{E}\left[|M_t| \mathbb{1}_{T \geq t}\right] \leq \mathbb{E}\left[\left(\max(a,b)^2 + T\right) \mathbb{1}\{T \geq t\}\right]$$

T has exponentially decaying tail

$$\Rightarrow \mathbb{E}\left[|M_t| \mathbb{1}_{T \geq t}\right] \to 0.$$

So by OST, we have

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T] = \mathbb{E}[B_T^2] - \mathbb{E}[T]$$

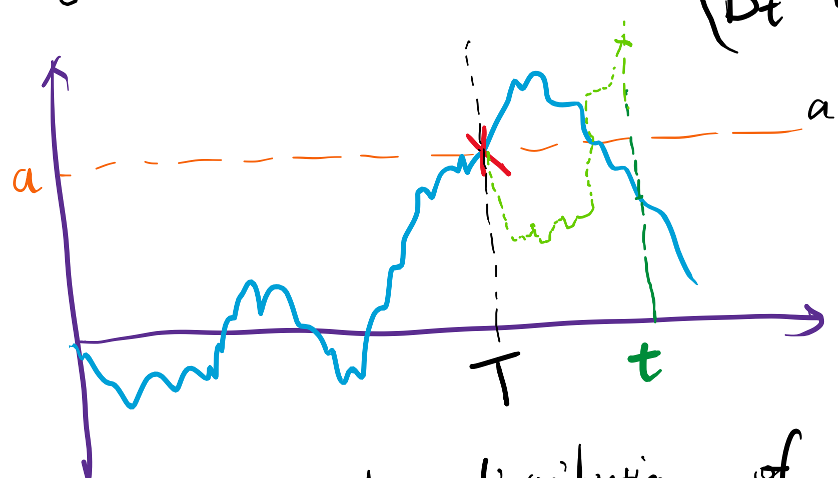
$$\text{So} \quad \mathbb{E}[T] = \mathbb{E}[B_T^2] = \frac{ba^2}{a+b} + \frac{ab^2}{a+b} = ab.$$

Reflection principle.

$(B_t)_{t \geq 0}$ BM,

$T := \inf\{t : B_t \geq a\}$ for some $a > 0$.

$(B_t - B_T)_{t \geq T}$ is a fresh new BM.



Can we get the distribution of T.

For any $t > 0$, want to compute

$$\mathbb{P}(T \leq t).$$

Intuition: after reaching T, two possibilities

$$\begin{cases} X_t \geq a \\ X_t < a \end{cases} \quad \text{same prob.}$$

probability $= \mathbb{P}(X_t \geq a)$.

Intuitively. $\mathbb{P}(T \leq t) = 2\mathbb{P}(X_t \geq a)$.

Making it rigorous:

By strong Markov property.

$$\mathbb{P}(X_t \geq a) = \mathbb{P}(T \leq t) \cdot \mathbb{P}(B_t \geq a \mid T \leq t).$$

$$= \mathbb{P}(T \leq t) \cdot \mathbb{P}(B_t - B_T \geq 0 \mid T \leq t)$$

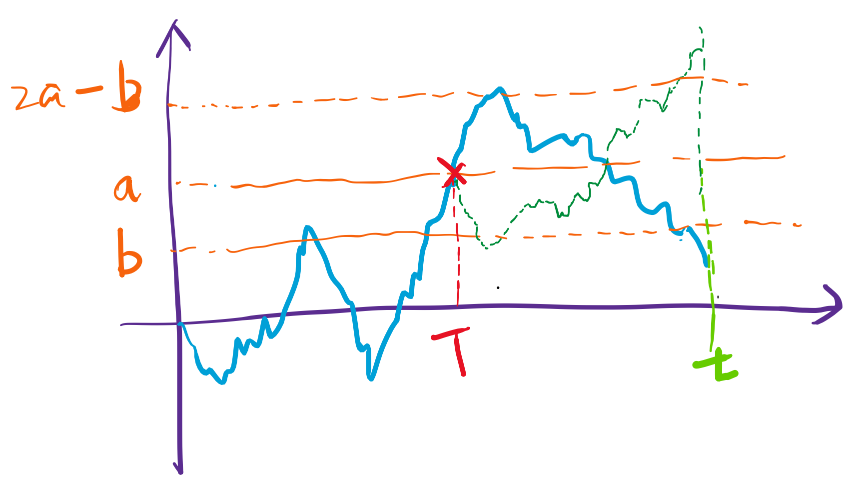
can condition on $(B_s)_{0 \leq s \leq T}$ for $T \leq t$.

$$= \frac{1}{2}\mathbb{P}(T \leq t).$$

$$\mathbb{P}(X_t \geq a) = \int_a^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right) dz$$

(can get density by taking derivative).

e.g.



T: stopping time

t: deterministic time.

$$\mathbb{P}(T \le t,\ B_t \le b) \quad (\text{for } b < a)$$

$$= \mathbb{P}(T \le t) \cdot \mathbb{P}(B_t \le b \mid T \le t)$$

$$= \mathbb{P}(T \le t) \cdot \mathbb{P}(B_t - B_T \le -(a-b) \mid T \le t)$$

$$= \mathbb{P}(T \le t) \cdot \mathbb{P}(B_t - B_T \ge a-b \mid T \le t)$$

$$= \mathbb{P}(T \le t,\ B_t \ge 2a-b)$$

$$= \mathbb{P}(B_t \ge 2a-b)$$

$$= \int_{2a-b}^{+\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{z^2}{2t}\right) dz.$$

Corollary. (T_a: hitting time of $a>0$)

$$\begin{array}{ccc} \mathbb{P}(T_a \le t) & = & 2 \cdot \mathbb{P}(B_t \ge a) \\ \| & & \| \\ \mathbb{P}\left(\max\limits_{0 \le s \le t} B_s \ge a\right) & & \mathbb{P}(|B_t| \ge a) \end{array}$$

So $\max\limits_{0 \le s \le t} B_s \overset{d}{=} |B_t|$.

Qualitive properties of BM.

Time derivative?

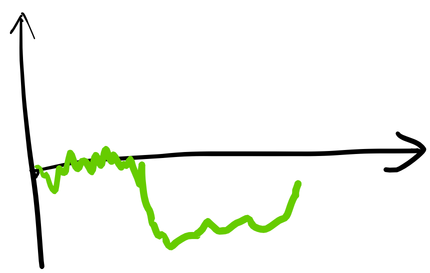
$$\frac{B_{t+\Delta t} - B_t}{\Delta t} \sim N\left(0, \frac{1}{\Delta t}\right)$$

Easy to show $\forall t,\ \mathbb{P}\left((B_s)_{s\geq 0} \text{ differentiable at } t\right) = 0.$

With some efforts, one can show

$$\mathbb{P}\left(\exists t \geq 0,\ B \text{ diff at } t\right) = 0.$$

Fact: $S := \{ t > 0 : B_t = 0 \}$



$\forall \varepsilon > 0,$

$S \cap [0, \varepsilon]$ is infinite