STA 447/2006 Lecture 9

From last lecture: L^1, L^p convergence of MGs.

This lecture: more examples.

e.g. Branching process.

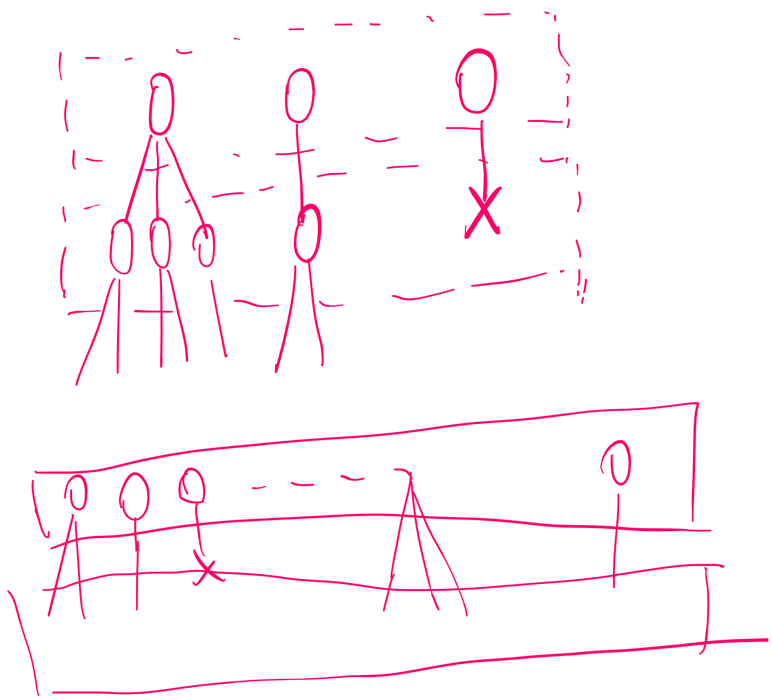
X_0

X_1

$\vdots$

X_n

X_{n+1}



Let μ be a prob. distr. on $\{0, 1, 2, \cdots\}$
"offspring distribution".

A branching process $(X_n)_{n \geq 0}$ is an MC defined by

$$X_{n+1} = Z_{n,1} + Z_{n,2} + \cdots + Z_{n,X_n}$$

where $Z_{n,i} \overset{iid}{\sim} \mu$ for $i=1,2,\cdots X_n$

(conditionally on X_n).

Key question: whether $(X_n)_{n\geq 0}$ dies out?

Assuming $\mathbb{E}[X_0] < +\infty$ throughout.

$$\mathbb{E}[X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1} \mid X_n]$$

$$= \sum_{i=1}^{X_n} \mathbb{E}[Z_{n,i}] = X_n \cdot \underbrace{\mathbb{E}_\mu[Z]}_{=: m}.$$

So $Y_n := m^{-n} \cdot X_n$ is an MG.

— When $m < 1$,

$$\mathbb{E}[X_n] = m^n \cdot \mathbb{E}[X_0] \longrightarrow 0 \quad (\text{as } n \to +\infty).$$

$$X_n \xrightarrow{L^1} 0 \quad (\text{which implies } X_n \xrightarrow{P} 0).$$

X_n is integer-valued, this implies a.s. convergence.

— When $m > 1$. $\mu(0) > 0$ (otherwise, impossible to die out)

- The process could still die out w.p. > 0

(e.g. die out in round 1 w.p. $\mu(0)^{X_0}$)

$\cdot\ \mathbb{E}[X_n] \longrightarrow +\infty.$

— $m=1$, $\mu(0)>0$

$(X_n)_{n\geq 1}$ is an MG

$X_n \geq 0$ So $\mathbb{E}[|X_n|] = \mathbb{E}[X_n] = \mathbb{E}[X_0] < +\infty.$

$X_n \xrightarrow{a.s.} X_\infty$ for some r.v. X_∞.

On the other hand, $(X_n)_{n\geq 0}$ takes only integer values

So $\exists\, T < +\infty$ such that after $n \geq T$, $X_n = X_\infty$.

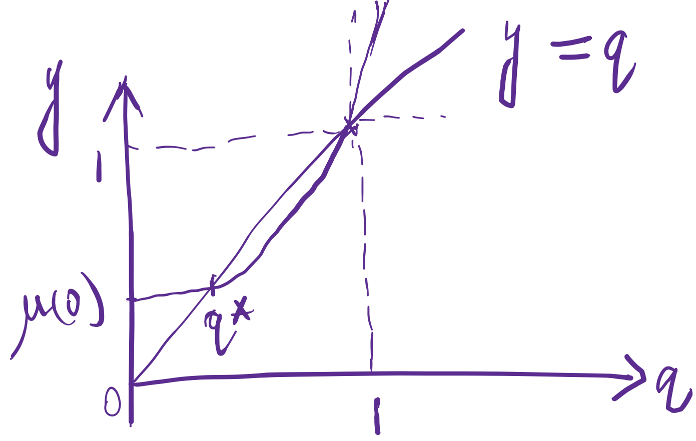
So $X_n \longrightarrow 0$ a.s.

Extinction prob. in super-critical case $(m>1)$.

$$\mathbb{P}(\text{Exctinction} \mid X_0 = k) = \mathbb{P}(\text{Exctinction} \mid X_0 = 1)^k.$$

$$q = \mathbb{P}(\text{Exctinction} \mid X_0 = 1).$$

$$q = \boxed{\sum_{k\geq 0} \mu(k) \cdot q^k}$$ generating func of μ.



Generating func is (strictly) convex, intersects w/ $y=q$ twice at $q=q^*$ and $q=1$.

When $m>1$, $\mathbb{P}(X_n \to +\infty) > 0$, and extinction prob. $= q^*$.

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e.g. Doob's MG.

X : a random variable $\mathbb{E}[|X|] < +\infty$.

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots \cdots \subseteq \mathcal{F}_n \subseteq \cdots\cdots$$

"flow of information".

$$M_n := \mathbb{E}[X \mid \mathcal{F}_n]$$

Claim: $(M_n)_{n\geq 0}$ is an MG.

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\big[\mathbb{E}[X \mid \mathcal{F}_{n+1}] \big| \mathcal{F}_n\big] = \mathbb{E}[X \mid \mathcal{F}_n]$$
$$= M_n.$$

Uniform integrability.

$$\mathbb{E}[|M_n| \mathbb{1}_{|M_n| \geq K}]$$
$$= \mathbb{E}\left[\left|\mathbb{E}[X|\mathcal{F}_n]\right| \cdot \mathbb{1}_{|M_n| \geq K}\right]$$
$$\leq \mathbb{E}\left[|X| \cdot \mathbb{1}_{\max_{1 \leq i \leq n} |M_i| \geq K}\right]. \quad \text{uniformly in } n \in \mathbb{N}.$$

By Doob's maximal ineq,

$$\mathbb{P}\left(\max_{1 \leq i \leq n} |M_i| \geq K\right) \leq \frac{\mathbb{E}|M_n|}{K} \leq \frac{\mathbb{E}|X|}{K} \longrightarrow 0.$$

$|X|$ as a dominating function, by DCT,

$$\mathbb{E}\left[|X| \cdot \mathbb{1}_{\max_{1 \leq i \leq n} |M_i| \geq K}\right] \longrightarrow 0 \qquad \text{So u.i.}$$

Consequently $\quad X_n \xrightarrow[L^1]{a.s.} X_\infty.$

(Indeed, $\mathcal{F}_\infty = \bigcup_{n=1}^{+\infty} \mathcal{F}_n$, "Info of the whole seq")

we have $X_\infty = \mathbb{E}[X|\mathcal{F}_\infty]$

If X is determined by the seq. then $X_\infty = X$.

Application: posterior consistency.

$$\theta \sim \pi \text{ (prior)}$$

$$X_1 \cdots X_n \mid \theta \overset{iid}{\sim} P_\theta$$

$$\pi(\theta \mid X_1 \cdots X_n) = \frac{\pi(\theta) \cdot P_\theta(X_1) P_\theta(X_2) \cdots P_\theta(X_n)}{\int \pi(\theta') P_{\theta'}(X_1) \cdots P_{\theta'}(X_n) \, d\theta'}.$$

Question: estimate $g(\theta)$

$$\hat{g}_n = \mathbb{E}[g(\theta) \mid X_1 \cdots X_n] \xrightarrow{?} g(\theta).$$

Let $X = g(\theta)$, $\mathcal{F}_n = (X_1, X_2, \cdots, X_n)$.

$(\hat{g}_n)_{n \geq 0}$ is Doob's MG.

$$\hat{g}_n \xrightarrow[L^1]{a.s.} g_\infty$$

(in Bayesian literature, $g_\infty = g(\theta)$ under mild conditions).