STA447/2006: Final Exam

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This exam contains 16 pages.

Total marks: 100 pts

Time Allowed: 180 minutes

Question 1. [20 points, 2.5 points each] Mark each statement with T (true) or F (false). No justification required.

(1) Let i, j be states in a Markov chain. If i is recurrent and $i \to j$, then j is recurrent. **Answer:** T

(2) Let P be an irreducible Markov chain. If P is null recurrent, then for any i, j pair, we have $\mathbb{E}_i[T_j] = +\infty$, where T_j denotes hitting time of j. **Answer:** F

(3) Let P be an irreducible Markov chain. Then P^2 is also irreducible. **Answer:** F

(4) If T_1 and T_2 are both stopping times, then $T := \max\big(T_1 + 2, T_2 + 3\big)$ is also a stopping time. **Answer:** T

(5) Let $(X_k)_{k=0,1,2,\cdots}$ be a martingale, satisfying $X_k \to X_\infty$ almost surely as $k \to \infty$. Then $(X_k)_{k\geq 0}$ is uniformly integrable. **Answer:** F

(6) For $i = 1, 2$, let $(B^{(i)}(t))_{t\geq 0}$ be independent standard Brownian motions. Then the process $(B^{(1)}(t) \cdot B^{(2)}(t))_{t\geq 0}$ is a martingale. **Answer:** T

(7) Let $(B(t))_{t\geq 0}$ be a standard Brownian motion, then $(B_t^2 - t)_{t\geq 0}$ is a martingale. **Answer:** T

(8) Let $(N_1(t))_{t\geq 0}$ be a Poisson process with intensity $\lambda = 1$. Let $(N_2(t))_{t\geq 0}$ be a Poisson process with intensity $\lambda = 2$. Assume that N_1 and N_2 are independent. Then the process $(N_2(t) - N_1(t))_{t\geq 0}$ is a Poisson process with intensity $\lambda = 1$. **Answer:** F

Question 2. [14 points] Consider a Markov chain on a finite state space $S = \{1, 2, 3, 4, 5\}$, with the transition matrix given by

$$P = \begin{bmatrix} 0 & 0 & 1/3 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \end{bmatrix}.$$

For $i \in S$, define the hitting time

$$T_i := \inf\big\{t \geq 1 : X_t = i\big\}.$$

(1) [7 pts]. Compute

$$\mathbb{E}\Big[\sum_{t=1}^{T_1} \mathbf{1}_{\{X_t=3\}} \mid X_0 = 1\Big]$$

i.e., the expected number of visits to state 3 before hitting state 1.

Answer: The chain is invariant with respect to permutations of the states $\{1, 2\}$ and $\{3, 4, 5\}$. Furthermore, we observe that it is reversible. So we have the stationary distribution

$$\pi = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

Invoking the excursion-based construction of the stationary measure (see Lecture 4, positive recurrence), we have

$$\mathbb{E}\Big[\sum_{t=1}^{T_1} \mathbf{1}_{\{X_t=3\}} \mid X_0 = 1\Big] = \frac{\pi_3}{\pi_1} = \frac{2}{3}.$$

Rubrics: 5 points for identifying a correct method (e.g. using stationary measure).

2 points for computing the answer correctly.

Alternatively, one can use a direct approach by writing f-expansion-like systems. Reasonable partial progress, such as a correct first-step system of equations, receives partial credit.

Any other correct method gets full credit.

(2) [7 pts]. Compute $\mathbb{P}_1(T_5 < T_2)$, i.e., the probability of hitting state 5 before hitting state 2, starting from state 1.

Answer: Let $q_i := \mathbb{P}_i(T_5 < T_2)$, we have the one-step expansion (similar to f-expansion)

$$\begin{aligned} q_1 &= \frac{1}{3}q_3 + \frac{1}{3}q_4 + \frac{1}{3}, \\ q_3 &= \frac{1}{2}q_1, \\ q_4 &= \frac{1}{2}q_1. \end{aligned}$$

Substituting the last two equations into the first,

$$q_1 = \frac{1}{3} \cdot \frac{q_1}{2} + \frac{1}{3} \cdot \frac{q_1}{2} + \frac{1}{3} = \frac{q_1}{3} + \frac{1}{3}.$$

Hence

$$q_1 = \frac{1}{2}.$$

Rubrics: 5 points for listing the correct system of equations.

2 points for solving the system correctly and obtaining the final answer 1/2.

This problem can also be solved by making states 2 and 5 absorbing and using the same first-step analysis. Any other correct method gets full credit, with partial progress receiving partial credit.

Question 3. [8 points] Let the state space be $S = \{0,1\}^D$ for some positive integer D. Consider a Markov chain on S with the following transition rule: at time t, suppose the current state is $X^{(t)} = x \in S$, we randomly pick a coordinate i uniformly from $\{1, 2, \cdots, D\}$, and then sample

$$B \sim \text{Bernoulli}\Big(\frac{1}{2} + \frac{1}{4}x_i\Big), \quad \text{for } i = 1, 2, \cdots, D,$$

and then update the state by replacing the i-th coordinate of x with B, while keeping the other coordinates unchanged.

Find the stationary distribution π of this Markov chain, and show that

$$\lim_{n\to+\infty} p^{(n)}_{x,y} = \pi_y, \quad \text{for any } x, y \in S,$$

Answer: For a single coordinate, the update rule is

$$0 \to 1 \text{ with probability } \frac{1}{2}, \qquad 1 \to 0 \text{ with probability } \frac{1}{4}.$$

Thus the one-coordinate transition matrix is

$$K = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}.$$

Its stationary law is Bernoulli(2/3), because if $p = \mathbb{P}(X_i = 1)$ then

$$p = \frac{1}{2}(1-p) + \frac{3}{4}p \qquad \Longleftrightarrow \qquad p = \frac{2}{3}.$$

Therefore the stationary distribution on $\{0,1\}^D$ is the product measure

$$\pi(x) = \prod_{i=1}^{D} \Big(\frac{2}{3}\Big)^{x_i} \Big(\frac{1}{3}\Big)^{1-x_i}, \qquad x = (x_1, \ldots, x_D) \in \{0,1\}^D.$$

To verify stationarity, it is enough to check detailed balance. If x and y differ in exactly one coordinate, say $x_i = 0$ and $y_i = 1$, then

$$P(x,y) = \frac{1}{D} \cdot \frac{1}{2}, \qquad P(y,x) = \frac{1}{D} \cdot \frac{1}{4},$$

and since $\pi(y) = 2\pi(x)$, we get

$$\pi(x)P(x,y) = \pi(y)P(y,x).$$

So π is reversible, hence stationary.

The chain is irreducible because each coordinate can change from 0 to 1 and from 1 to 0 with positive probability. It is aperiodic because every state has a positive self-loop probability:

$$P(x,x) = \frac{\#\{i : x_i = 0\}}{D} \cdot \frac{1}{2} + \frac{\#\{i : x_i = 1\}}{D} \cdot \frac{3}{4} > 0.$$

Since the state space is finite, irreducibility and aperiodicity imply

$$\lim_{n\to\infty} p_{x,y}^{(n)} = \pi(y), \qquad \forall x, y \in S.$$

Rubrics: 4 points for identifying the correct stationary distribution, namely the product Bernoulli(2/3) law.

2 points for justifying stationarity, for example by detailed balance.

2 points for proving convergence by showing the chain is finite, irreducible, and aperiodic.

Any other correct argument gets full credit.

Question 4. [28 pts] Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(1) [7 pts]. Compute the quantity

$$\mathbb{E}\Big[\big|\int_0^1 B_t dB_t\big|^2\Big]$$

The answer should be in closed form (i.e., no limits or integrals in your final expression).

Answer: By Itô's isometry, we have

$$\mathbb{E}\Big[\big|\int_0^1 B_t dB_t\big|^2\Big] = \int_0^1 \mathbb{E}[B_t^2]dt = \int_0^1 t dt = \frac{1}{2}.$$

Rubrics: Alternatively, you can also compute the Itô integral as $(B_1^2 - 1)/2$ and use brute-force.

5 points for identifying the correct method (e.g. using Itô's isometry or computing the integral explicitly).

2 points for computing the final answer $\frac{1}{2}$ correctly.

Partial progress also receives partial credit.

(2) [7 pts]. Suppose that $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ satisfies

$$dX_t = B_t dB_t, \quad dY_t = X_t dB_t.$$

Find a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the process

$$M_t = X_t Y_t - \int_0^t h(B_s, X_s) ds$$

is a martingale, and compute $\mathbb{E}[h(B_t, X_t)]$.

Answer: Apply Itô's product rule:

$$\begin{aligned} d(X_t Y_t) &= X_t\, dY_t + Y_t\, dX_t + d[X, Y]_t \\ &= X_t(X_t\, dB_t) + Y_t(B_t\, dB_t) + B_t X_t\, dt \\ &= (X_t^2 + B_t Y_t)\, dB_t + B_t X_t\, dt. \end{aligned}$$

Therefore we may take

$$h(b, x) = bx.$$

Then

$$d\Big(X_t Y_t - \int_0^t B_s X_s\, ds\Big) = (X_t^2 + B_t Y_t)\, dB_t,$$

so the process is a martingale.

Next,

$$X_t = \int_0^t B_s\, dB_s = \frac{B_t^2 - t}{2}$$

by Itô's formula. Hence

$$\mathbb{E}[h(B_t, X_t)] = \mathbb{E}[B_t X_t] = \frac{1}{2}\mathbb{E}[B_t^3 - tB_t] = 0,$$

because B_t is centered Gaussian and all odd moments vanish.

Rubrics: 4 points for applying Itô's product rule correctly and identifying the drift term $B_t X_t$.

3 points for computing $\mathbb{E}[h(B_t, X_t)] = 0$ correctly.

Any other correct method gets full credit.

(3) [7 pts]. Let $T_1 := \inf\{t \geq 0 : B_t \geq 1\}$. Compute the conditional probability

$$\mathbb{P}(T_1 \leq 1 \mid B_1 \leq 0).$$

You can express your answer in terms of the cumulative distribution function of $\mathcal{N}(0,1)$, but no integrals or limits should appear in your final answer.

Answer: Let

$$A = \{T_1 \leq 1,\ B_1 \leq 0\}.$$

By the reflection principle, reflecting a Brownian path at its first hitting time of level 1 gives a bijection between the events

$$\{T_1 \leq 1,\ B_1 \leq 0\} \quad \text{and} \quad \{B_1 \geq 2\}.$$

Hence

$$\mathbb{P}(T_1 \leq 1,\ B_1 \leq 0) = \mathbb{P}(B_1 \geq 2) = 1 - \Phi(2).$$

Also,

$$\mathbb{P}(B_1 \leq 0) = \frac{1}{2}.$$

Therefore

$$\mathbb{P}(T_1 \leq 1 \mid B_1 \leq 0) = \frac{\mathbb{P}(T_1 \leq 1,\ B_1 \leq 0)}{\mathbb{P}(B_1 \leq 0)} = 2\big(1 - \Phi(2)\big).$$

Rubrics: 4 points for identifying the joint event $\{T_1 \leq 1, B_1 \leq 0\}$ and applying the reflection principle correctly.

3 points for dividing by $\mathbb{P}(B_1 \leq 0) = 1/2$ and obtaining the final answer $2(1 - \Phi(2))$.

Any other correct method gets full credit.

(4) [7 pts]. Define the process

$$M_t := \exp\big(B_t - \frac{1}{2}t\big).$$

Show that M_t is a martingale, and use this fact to compute the probability

$$\mathbb{P}\Big(\max_{t\geq 0} M_t \geq 2\Big).$$

Answer: By Itô's formula,

$$dM_t = M_t\, d\Big(B_t - \frac{t}{2}\Big) + \frac{1}{2}M_t\, d\langle B\rangle_t = M_t\, dB_t.$$

Therefore M_t is a martingale.

Let

$$\tau := \inf\{t \geq 0 : M_t \geq 2\}.$$

Since M_t is continuous, we have $M_\tau = 2$ on $\{\tau \leq t\}$, while $M_t < 2$ on $\{\tau > t\}$. Hence $0 \leq M_{t\wedge\tau} \leq 2$, so optional stopping applies to the bounded martingale $M_{t\wedge\tau}$:

$$1 = M_0 = \mathbb{E}[M_{t\wedge\tau}] = 2\mathbb{P}(\tau \leq t) + \mathbb{E}[M_t \mathbf{1}_{\{\tau>t\}}].$$

Now $B_t/t \to 0$ almost surely, so

$$M_t = \exp\Big(B_t - \frac{t}{2}\Big) \to 0 \qquad \text{almost surely as } t \to \infty.$$

Since $0 \leq M_t \mathbf{1}_{\{\tau>t\}} \leq 2$, dominated convergence gives

$$\mathbb{E}[M_t \mathbf{1}_{\{\tau>t\}}] \to 0.$$

Letting $t \to \infty$ in the optional stopping identity yields

$$1 = 2\mathbb{P}(\tau < \infty).$$

Therefore

$$\mathbb{P}\Big(\max_{t\geq 0} M_t \geq 2\Big) = \mathbb{P}(\tau < \infty) = \frac{1}{2}.$$

Rubrics: 3 points for showing correctly that M_t is a martingale.

4 points for introducing the hitting time, applying optional stopping, and obtaining the correct probability $1/2$.

The two parts of the credit are awarded independently. A correct calculation for the probability without justifying martingale still receives the latter part of points.

Question 5. [13 pts] Consider a continuous-time discrete-space Markov chain on $\{1, 2, \cdots\}$, with the generator given by

$$g_{i,i} = -\lambda_i, \quad g_{i,i+1} = \lambda_i, \quad \text{for } i = 1, 2, \cdots,$$

Define the stopping times

$$T_n := \inf\{t \geq 0 : X_t = n\}, \quad T_\infty := \lim_{n \to +\infty} T_n.$$

(1) [6 pts]. Suppose that $\lambda_i = i^2$ for $i = 1, 2, \cdots$. Show that

$$\sup_{n \geq 1} \mathbb{E}_1[T_n] < +\infty,$$

and conclude that $T_\infty < +\infty$ almost surely.

Answer: Let E_i be the holding time spent in state i. Then

$$E_i \sim \text{Exp}(i^2), \qquad \mathbb{E}[E_i] = \frac{1}{i^2},$$

and the holding times are independent. Therefore

$$T_n = \sum_{i=1}^{n-1} E_i,$$

so

$$\mathbb{E}_1[T_n] = \sum_{i=1}^{n-1} \frac{1}{i^2} \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Hence

$$\sup_{n \geq 1} \mathbb{E}_1[T_n] \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

Since $T_n \uparrow T_\infty$ almost surely, monotone convergence gives

$$\mathbb{E}_1[T_\infty] = \lim_{n \to \infty} \mathbb{E}_1[T_n] \leq \sum_{i=1}^{\infty} \frac{1}{i^2} < \infty.$$

If $\mathbb{P}_1(T_\infty = \infty) > 0$, then $\mathbb{E}_1[T_\infty] = \infty$, which is impossible. Therefore

$$T_\infty < \infty \qquad \text{almost surely.}$$

Rubrics: 4 points for decomposing T_n into a sum of independent exponential holding times and computing $\mathbb{E}_1[T_n]$.

2 points for concluding correctly that $T_\infty < \infty$ almost surely.

Any other correct method gets full credit.

(2) [7 pts]. Suppose that $\lambda_i = i$ for $i = 1, 2, \cdots$. Show that

$$\mathbb{E}_1\left[e^{-T_n}\right] \to 0, \quad \text{as } n \to +\infty,$$

and conclude that $T_\infty = +\infty$ almost surely.

Answer: Again write

$$T_n = \sum_{i=1}^{n-1} E_i,$$

where $E_i \sim \text{Exp}(i)$ are independent. For an exponential random variable with rate i,

$$\mathbb{E}[e^{-E_i}] = \int_0^\infty e^{-t} i e^{-it}\, dt = \frac{i}{i+1}.$$

Hence

$$\mathbb{E}_1[e^{-T_n}] = \prod_{i=1}^{n-1} \mathbb{E}[e^{-E_i}] = \prod_{i=1}^{n-1} \frac{i}{i+1} = \frac{1}{n} \to 0.$$

Since $T_n \uparrow T_\infty$ almost surely, we have $e^{-T_n} \to e^{-T_\infty}$ almost surely. Also $0 \le e^{-T_n} \le 1$, so dominated convergence gives

$$\mathbb{E}_1[e^{-T_\infty}] = \lim_{n\to\infty} \mathbb{E}_1[e^{-T_n}] = 0.$$

But $e^{-T_\infty} > 0$ on the event $\{T_\infty < \infty\}$. Therefore

$$\mathbb{P}_1(T_\infty < \infty) = 0,$$

that is,

$$T_\infty = \infty \qquad \text{almost surely.}$$

Rubrics: 4 points for writing T_n as a sum of independent exponential holding times and computing the Laplace transform correctly.

3 points for concluding from $\mathbb{E}_1[e^{-T_\infty}] = 0$ that $T_\infty = \infty$ almost surely.

Any other correct method gets full credit.

Question 6. [17 pts] Expected hitting times

(1) [6 pts]. Consider a Markov chain on state space S, and let $A \subset S$ be a subset of states. Suppose that the function f satisfies

$$f(x) = 0, \quad \text{for } x \in A,$$
$$f(x) = 1 + \sum_{y \in S} P(x,y) f(y), \quad \text{for } x \in S \setminus A.$$

Show that $f(x) = \mathbb{E}_x[T_A]$, where T_A is the hitting time of set A.

Answer: Define

$$M_n := f(X_{n \wedge T_A}) + (n \wedge T_A).$$

We claim that (M_n) is a martingale. If $n \geq T_A$, then $M_{n+1} = M_n$. If $n < T_A$, then $X_n \notin A$, so by the assumed recursion,

$$\mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] = \sum_{y \in S} P(X_n, y) f(y) = f(X_n) - 1.$$

Therefore on $\{n < T_A\}$,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[f(X_{n+1}) \mid \mathcal{F}_n] + (n+1) = f(X_n) + n = M_n.$$

So (M_n) is a martingale.

Taking expectations gives

$$f(x) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_n] = \mathbb{E}_x\big[f(X_{n \wedge T_A}) + (n \wedge T_A)\big].$$

Since $f = 0$ on A, we have $f(X_{T_A}) = 0$. Letting $n \to \infty$ yields

$$f(x) = \mathbb{E}_x[T_A].$$

This is the desired identity.

Rubrics: 3 points for constructing the martingale $f(X_{n \wedge T_A}) + n \wedge T_A$ or an equivalent first-step argument.

3 points for taking expectations / applying optional stopping and concluding $f(x) = \mathbb{E}_x[T_A]$.

Any other correct method gets full credit.

You still receive full credit if you prove the finite statespace case.

(2) [6 pts]. Let $S = \{0, 1, \cdots, 2m+1\}$ and $A = \{0, 2m+1\}$. Consider a Markov chain on S with transition matrix given by

$$p_{x,x+1} = \begin{cases} \frac{1}{4} & \text{if } x \in \{0, 1, \cdots, m-1\} \\ \frac{1}{2} & \text{if } x \in \{m, m+1\} \\ \frac{3}{4} & \text{if } x \in \{m+2, m+3, \cdots, 2m\}, \end{cases}$$

and $p_{x,x-1} = 1 - p_{x,x+1}$ for $x \in \{1, 2, \cdots, 2m\}$, and $p_{0,0} = p_{2m+1,2m+1} = 1$. Compute $\mathbb{E}_x[T_A]$ for $x \in \{1, 2, \cdots, 2m\}$. You can use the result from part (1) even if you did not prove it.

Answer: Let

$$f_x := \mathbb{E}_x[T_A], \qquad x = 0, 1, \ldots, 2m+1.$$

Then $f_0 = f_{2m+1} = 0$, and by part (1), for $1 \le x \le 2m$,

$$f_x = 1 + p_{x,x+1} f_{x+1} + p_{x,x-1} f_{x-1}.$$

That is,

$$\begin{aligned} f_x &= 1 + \frac{1}{4} f_{x+1} + \frac{3}{4} f_{x-1}, && 1 \le x \le m-1, \\ f_x &= 1 + \frac{1}{2} f_{x+1} + \frac{1}{2} f_{x-1}, && x = m, m+1, \\ f_x &= 1 + \frac{3}{4} f_{x+1} + \frac{1}{4} f_{x-1}, && m+2 \le x \le 2m. \end{aligned}$$

Define the increments $d_x = f_x - f_{x-1}$ for $x = 1, \ldots, 2m+1$. Then the recurrence becomes

$$\begin{aligned} d_{x+1} &= 3d_x - 4, && 1 \le x \le m-1, \\ d_{x+1} &= d_x - 2, && x = m, m+1, \\ d_{x+1} &= \frac{d_x - 4}{3}, && m+2 \le x \le 2m. \end{aligned}$$

By symmetry of the chain under $x \mapsto 2m+1-x$, we have $f_x = f_{2m+1-x}$, hence $d_{m+1} = f_{m+1} - f_m = 0$. Using the recurrences,

$$\begin{aligned} d_x &= 2, && 1 \le x \le m, \\ d_{m+1} &= 0, && \\ d_x &= -2, && m+2 \le x \le 2m+1. \end{aligned}$$

Therefore

$$f_x = \sum_{k=1}^{x} d_k = \begin{cases} 2x, & 1 \le x \le m, \\ 2(2m+1-x), & m+1 \le x \le 2m. \end{cases}$$

Equivalently,

$$\mathbb{E}_x[T_A] = 2\min\{x, 2m+1-x\}, \qquad x = 1, 2, \ldots, 2m.$$

Rubrics: 3 points for setting up the correct recurrence for $f_x = \mathbb{E}_x[T_A]$.
3 points for solving the recurrence and obtaining $\mathbb{E}_x[T_A] = 2\min\{x, 2m+1-x\}$.
Any other correct method gets full credit.

(3) [5 pts]. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion. Consider the process

$$dX_t = (X_t - 1)dt + dB_t.$$

Define the stopping time

$$T := \inf\{t \geq 0 : X_t \in \{0, 2\}\}.$$

Suppose that the function f solves the ordinary differential equation

$$(x-1)f'(x) + \frac{1}{2}f''(x) = -1, \quad \text{for } x \in (0,2),$$

with boundary condition $f(0) = f(2) = 0$. Show that $f(x) = \mathbb{E}_x[T]$ for $x \in (0,2)$. You do not need to solve the ODE explicitly.

Answer: Apply Itô's formula to $f(X_t)$ up to the stopping time T. For $t \leq T$,

$$\begin{aligned} df(X_t) &= f'(X_t)\,dX_t + \frac{1}{2}f''(X_t)\,dt \\ &= f'(X_t)\big((X_t-1)dt + dB_t\big) + \frac{1}{2}f''(X_t)\,dt \\ &= \Big((X_t-1)f'(X_t) + \frac{1}{2}f''(X_t)\Big)dt + f'(X_t)\,dB_t \\ &= -dt + f'(X_t)\,dB_t. \end{aligned}$$

Hence

$$f(X_{t\wedge T}) + (t \wedge T)$$

is a martingale. Taking expectation under $\mathbb{P}_x$ gives

$$f(x) = \mathbb{E}_x\big[f(X_{t\wedge T}) + (t\wedge T)\big].$$

Now let $t \to \infty$. Since $X_T \in \{0,2\}$ and $f(0) = f(2) = 0$, we have $f(X_T) = 0$. Therefore

$$f(x) = \mathbb{E}_x[T].$$

This proves the claim.

Rubrics: 2 points for applying Itô's formula correctly and obtaining the drift term -1.

3 points for constructing the martingale and concluding that $f(x) = \mathbb{E}_x[T]$.

Any other correct method gets full credit.

You may assume existence/uniqueness of the solution to the ODE and still get full credit.