STA447/2006: Midterm Exam #2

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This exam contains 11 pages.

Total marks: 100 pts

Time Allowed: 150 minutes

Question 1. [30 points, 3 points for each question] Mark each of the following statements with T (true) or F (false). *No justification is required.* Your grade will be solely based on your true-or-false choices.

(1) For a discrete-time stochastic process $(X_n)_{n=0,1,2,\cdots}$, if T_1 and T_2 are both stopping times satisfying $T_1 \geq T_2$, then $T_1 - T_2$ is a stopping time. **Answer:** F

(2) Let T_1, T_2, T_3 be three stopping times. Let $T_{(2)}$ be the second smallest one among T_1, T_2, T_3. Then $T_{(2)}$ is a stopping time. **Answer:** T

(3) Let $T := \min\{t \geq 10 : X_t = X_{t-10}\}$. The random time T is a stopping time. **Answer:** T

(4) If $(Z_i)_{i\geq 0}$ are i.i.d. zero-mean random variables, the following process $(X_n)_{n\geq 1}$ is a martingale

$$X_n := \Big(\sum_{i=1}^{n} Z_i\Big)^2 - n\mathbb{E}[Z_0^2].$$

Answer: T

(5) Let $(X_n)_{n\geq 0}$ be a discrete-time martingale such that $X_n \geq 0$ for all n. Then there exists a random variable X_∞ such that $X_n \to X_\infty$ a.s. **Answer:** T

(6) Let $(X_n)_{n=0,1,2\cdots}$ be a discrete-time martingale, if for any $\varepsilon > 0$ and any $n \geq 0$, there exists a constant $K > 0$ depending on ε and n such that

$$\mathbb{E}\Big[|X_n|\mathbf{1}_{|X_n|\geq K}\Big] \leq \varepsilon,$$

then the process $(X_n)_{n\geq 0}$ is uniformly integrable. **Answer:** F

(7) Let $(X_n)_{n=0,1,2\cdots}$ be a martingale. If there exists a stopping time such that $\mathbb{E}[X_T] = E[X_0]$, then $(X_n)_{n\geq 0}$ is uniformly integrable. **Answer:** F

(8) Let $X_n := \mathbb{E}[X|\mathcal{F}_n]$ for some random variable X and some filtration $(\mathcal{F}_n)_{n\geq 0}$, and T is a stopping time. Then X_T is well-defined and $\mathbb{E}[X_T] = \mathbb{E}[X_0]$. **Answer:** T

(9) Let $Z_1, Z_2, \cdots$ be i.i.d. random variables with finite mean μ. Let $X_n := \sum_{i=1}^{n} Z_i$. Let $T := \arg\max_{1\leq n\leq 10} X_n$. Then $\mathbb{E}[X_T] = \mu\mathbb{E}[T]$. **Answer:** F

(10) Let $(X_n)_{n\geq 0}$ be a martingale satisfying $\sup_{n\geq 0}\mathbb{E}[|X_n|^2] < +\infty$. Then we have $\mathbb{E}\big[\sup_{n\geq 0}|X_n|^2\big] < +\infty$. **Answer:** T

Rubrics: Each item is worth 3 points. No justification is required, and no partial credit is given.

Question 2. [24 points] Let $X_0 = 1$ and $X_t = W_t X_{t-1}$ where W_t are i.i.d. random variables, following the distribution

$$W_t = \begin{cases} \frac{3}{2} & \text{with probability } 1/2, \\ \frac{1}{2} & \text{with probability } 1/2, \end{cases}$$

and the random time $T_a := \inf\{t > 0 : X_t \geq a\}$ for some constant $a > 1$.

Part (a) [9 pt]. Is $(X_t)_{t\geq 0}$ uniformly integrable? How about the stopped process $(X_{t\wedge T_a})_{t\geq 0}$? Justify your answer.

Answer: Since $\mathbb{E}[W_t] = 1$, we have

$$\mathbb{E}[X_t \mid \mathcal{F}_{t-1}] = X_{t-1}\mathbb{E}[W_t] = X_{t-1},$$

so $(X_t)_{t\geq 0}$ is a nonnegative martingale with $\mathbb{E}[X_t] = 1$ for all t.

However,

$$\log X_t = \sum_{i=1}^{t} \log W_i,$$

and by the strong law of large numbers,

$$\frac{1}{t}\log X_t \to \frac{1}{2}\log\frac{3}{2} + \frac{1}{2}\log\frac{1}{2} = \frac{1}{2}\log\frac{3}{4} < 0 \qquad \text{a.s.}$$

Therefore $X_t \to 0$ almost surely. If (X_t) were uniformly integrable, then by the martingale convergence theorem we would have

$$\mathbb{E}[X_\infty] = \lim_{t\to\infty} \mathbb{E}[X_t] = 1,$$

but $X_\infty = 0$ almost surely, a contradiction. Hence $(X_t)_{t\geq 0}$ is *not* uniformly integrable.

For the stopped process, note that if $t < T_a$, then $X_t < a$, while on the event $T_a < +\infty$,

$$X_{T_a} = W_{T_a} X_{T_a - 1} \leq \frac{3}{2}a$$

because $X_{T_a-1} < a$ and $W_{T_a} \leq 3/2$. Hence

$$0 \leq X_{t\wedge T_a} \leq \frac{3}{2}a \qquad \text{for all } t.$$

So $(X_{t\wedge T_a})_{t\geq 0}$ is a bounded martingale, and therefore it is uniformly integrable.

Rubrics: 3 points for showing that $X_t \to 0$ almost surely, or giving an equivalent argument based on $\sum_{i=1}^{t} \log W_i$.

3 points for concluding that $(X_t)_{t\geq 0}$ is not uniformly integrable.

3 points for showing that $(X_{t\wedge T_a})_{t\geq 0}$ is bounded, hence uniformly integrable.

Part (b) [9 pt]. Show that

$$\frac{2}{3a} \leq \mathbb{P}(T_a < +\infty) \leq \frac{1}{a}$$

Answer: By part (a), the stopped process $(X_{t\wedge T_a})_{t\geq 0}$ is uniformly integrable, so we can invoke the optional stopping theorem for possibly infinite stopping times (from lecture) to obtain that

$$\mathbb{E}[X_{T_a}] = \mathbb{E}[X_0] = 1,$$

which can be rewritten as

$$1 = \mathbb{E}[X_{T_a}\mathbf{1}_{T_a<+\infty}] + \mathbb{E}[X_{T_a}\mathbf{1}_{T_a=+\infty}] = \mathbb{E}\big[X_{T_a} \mid T_a < +\infty\big] \cdot \mathbb{P}(T_a < +\infty),$$

since $X_{T_a} = \lim_{t\to\infty} X_{t\wedge T_a} = 0$ on the event $\{T_a = +\infty\}$.

On the event $\{T_a < +\infty\}$, we have

$$a \leq X_{T_a} \leq \frac{3}{2}a.$$

So we have

$$a \leq \mathbb{E}\big[X_{T_a} \mid T_a < +\infty\big] \leq \frac{3}{2}a.$$

Substituting this bound into the above equation gives

$$\frac{2}{3a} \leq \mathbb{P}(T_a < +\infty) \leq \frac{1}{a}.$$

Rubrics: 4 points for correctly applying optional stopping to get $\mathbb{E}[X_{T_a}] = 1$.

If you get the correct equation but without justifying the use of optional stopping, you can still get 2 points for this part.

5 points for using $a \leq X_{T_a} \leq 3a/2$ on $\{T_a < +\infty\}$ to derive both bounds.

If only one side of the bound is proved correctly, you get 3 points out of 5 for this part.

Part (c) [6 pt]. Define the random variable

$$Z := \max_{t \geq 0} X_t.$$

Use the result in part (a) to show that

$$\mathbb{P}(Z < +\infty) = 1, \quad \text{but} \quad \mathbb{E}[Z] = +\infty.$$

You can use the result in part (b) even if you cannot solve part (b).

Answer: From part (a), we know that $X_t \to 0$ almost surely. Hence on every sample path there exists a random time N such that $X_t \leq 1$ for all $t \geq N$. Therefore

$$Z = \max_{t \geq 0} X_t = \max \left\{ X_0, X_1, \ldots, X_{N-1}, \sup_{t \geq N} X_t \right\}$$

is finite almost surely. Thus

$$\mathbb{P}(Z < +\infty) = 1.$$

On the other hand, for every $a > 1$,

$$\{Z \geq a\} = \{T_a < +\infty\},$$

so part (b) gives

$$\mathbb{P}(Z \geq a) = \mathbb{P}(T_a < +\infty) \geq \frac{2}{3a}.$$

Using the tail-sum lower bound for nonnegative random variables,

$$\mathbb{E}[Z] \geq \sum_{k=2}^{\infty} \mathbb{P}(Z \geq k) \geq \sum_{k=2}^{\infty} \frac{2}{3k} = +\infty.$$

Therefore Z is finite almost surely, but $\mathbb{E}[Z] = +\infty$.

Rubrics: 3 points for proving $Z < +\infty$ almost surely, for example from $X_t \to 0$ a.s.

3 points for proving $\mathbb{E}[Z] = +\infty$ using the lower tail bound from part (b) and a tail-sum argument.

Question 3. [21 points] Let $(X_n)_{n\geq 0}$ be a one-dimensional symmetric simple random walk, i.e., $X_0 = 0$ and $X_n = \sum_{i=1}^{n} Z_i$ where Z_i are i.i.d. random variables such that $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = -1) = 1/2$.

For $\theta \in \mathbb{R}$, define $\lambda(\theta) := \log\left(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{-\theta}\right)$. We can construct a process $(M_n(\theta))_{n\geq 0}$ as follows:

$$M_n(\theta) := e^{\theta X_n - n\lambda(\theta)}.$$

Part (a) [4 points]. Prove that the process $(M_n(\theta))_{n\geq 0}$ is a martingale, for any $\theta \in \mathbb{R}$.

Answer: Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n)$. Since $X_n = X_{n-1} + Z_n$, we have

$$\begin{aligned}
\mathbb{E}[M_n(\theta) \mid \mathcal{F}_{n-1}] &= \mathbb{E}\big[e^{\theta X_n - n\lambda(\theta)} \mid \mathcal{F}_{n-1}\big] \\
&= e^{\theta X_{n-1} - n\lambda(\theta)} \mathbb{E}[e^{\theta Z_n} \mid \mathcal{F}_{n-1}] \\
&= e^{\theta X_{n-1} - n\lambda(\theta)} \left(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{-\theta}\right) \\
&= e^{\theta X_{n-1} - n\lambda(\theta)} e^{\lambda(\theta)} \\
&= e^{\theta X_{n-1} - (n-1)\lambda(\theta)} = M_{n-1}(\theta).
\end{aligned}$$

Therefore $(M_n(\theta))_{n\geq 0}$ is a martingale for every $\theta \in \mathbb{R}$.

Rubrics: 2 points for conditioning on $\mathcal{F}_{n-1}$ and using $X_n = X_{n-1} + Z_n$.
2 points for computing $\mathbb{E}[e^{\theta Z_n}] = e^{\lambda(\theta)}$ and concluding the martingale property.

Part (b) [9 points]. For a positive integer a, define the stopping time

$$T := \inf\{n > 0 : |X_n| \geq a\}.$$

Use the optional stopping theorem to show that

$$\mathbb{E}\big[e^{-\lambda(\theta)T}\big] = \frac{2}{e^{\theta a} + e^{-\theta a}}.$$

[Hint: You may first show that X_T is independent of T and use this fact to compute $\mathbb{E}[M_T(\theta)]$. If you cannot show the independence, you can still get partial credit by proving the desired result by assuming the independence.]

Answer: Let $\lambda = \lambda(\theta)$. Since T is the first time the walk exits $\{-a+1, \ldots, a-1\}$, we have $|X_{n\wedge T}| \leq a$ for every n. Hence

$$0 \leq M_{n\wedge T}(\theta) = e^{\theta X_{n\wedge T} - (n\wedge T)\lambda} \leq e^{|\theta| a},$$

so $(M_n(\theta))_{n\geq 0}$ is bounded up to time T. By optional stopping,

$$\mathbb{E}[M_T(\theta)] = \mathbb{E}[M_0(\theta)] = 1.$$

Next we show that X_T is independent of T. Since the walk moves by ± 1 and a is an integer, we must have $X_T \in \{-a, a\}$. For any fixed n, there is a sign-flip bijection between paths of length n with $T = n$ and $X_T = a$, and paths of length n with $T = n$ and $X_T = -a$. Therefore

$$\mathbb{P}(T = n, X_T = a) = \mathbb{P}(T = n, X_T = -a) = \frac{1}{2}\mathbb{P}(T = n).$$

Thus X_T is independent of T, and

$$\mathbb{P}(X_T = a) = \mathbb{P}(X_T = -a) = \frac{1}{2}.$$

Now compute

$$\begin{aligned} 1 = \mathbb{E}[M_T(\theta)] &= \mathbb{E}\big[e^{\theta X_T - \lambda T}\big] \\ &= \mathbb{E}[e^{-\lambda T}]\mathbb{E}[e^{\theta X_T}] \\ &= \mathbb{E}[e^{-\lambda T}] \cdot \frac{e^{\theta a} + e^{-\theta a}}{2}. \end{aligned}$$

Therefore,

$$\mathbb{E}[e^{-\lambda T}] = \frac{2}{e^{\theta a} + e^{-\theta a}}.$$

This is the desired identity.

Rubrics: 3 points for correctly applying optional stopping to the stopped martingale.

3 points for showing that X_T is independent of T so that $\mathbb{P}(X_T = \pm a) = 1/2$.

3 points for computing $\mathbb{E}[M_T(\theta)]$ and deriving the stated formula.

Part (c) [8 points]. Fix $\delta > 0$ and $n \in \mathbb{N}_+$, show that

$$\mathbb{P}\Big(\max_{0\le t\le n} X_t \le \sqrt{2n\log(1/\delta)}\Big) \ge 1-\delta.$$

Hint: you may use without proof the calculus fact that $\frac{1}{2}e^{\theta} + \frac{1}{2}e^{-\theta} \le e^{\theta^2/2}$ for any $\theta \in \mathbb{R}$. The martingale constructed in part (a) may be useful for this question.

Answer: Fix $\theta > 0$ and $b > 0$. If $\max_{0\le t\le n} X_t \ge b$, then for some $t \le n$ we have

$$M_t(\theta) = e^{\theta X_t - t\lambda(\theta)} \ge e^{\theta b - n\lambda(\theta)},$$

because $t \le n$ and $\lambda(\theta) \ge 0$. Therefore

$$\Big\{\max_{0\le t\le n} X_t \ge b\Big\} \subseteq \Big\{\max_{0\le t\le n} M_t(\theta) \ge e^{\theta b - n\lambda(\theta)}\Big\}.$$

Since $(M_t(\theta))_{0\le t\le n}$ is a nonnegative martingale, Doob's maximal inequality gives

$$\begin{aligned}\mathbb{P}\Big(\max_{0\le t\le n} X_t \ge b\Big) &\le \mathbb{P}\Big(\max_{0\le t\le n} M_t(\theta) \ge e^{\theta b - n\lambda(\theta)}\Big)\\ &\le e^{-\theta b + n\lambda(\theta)}\mathbb{E}[M_n(\theta)]\\ &= e^{-\theta b + n\lambda(\theta)}.\end{aligned}$$

Using the given inequality,

$$\lambda(\theta) = \log\Big(\frac{1}{2}e^{\theta} + \frac{1}{2}e^{-\theta}\Big) \le \frac{\theta^2}{2},$$

so

$$\mathbb{P}\Big(\max_{0\le t\le n} X_t \ge b\Big) \le e^{-\theta b + n\theta^2/2}.$$

Choose $\theta = b/n$. Then

$$\mathbb{P}\Big(\max_{0\le t\le n} X_t \ge b\Big) \le e^{-b^2/(2n)}.$$

Now set

$$b = \sqrt{2n\log(1/\delta)}.$$

Then

$$\mathbb{P}\Big(\max_{0\le t\le n} X_t \ge \sqrt{2n\log(1/\delta)}\Big) \le \delta,$$

which is equivalent to

$$\mathbb{P}\Big(\max_{0\le t\le n} X_t \le \sqrt{2n\log(1/\delta)}\Big) \ge 1-\delta.$$

Rubrics: 4 points for reducing the event $\{\max_{t\le n} X_t \ge b\}$ to a maximal event for the martingale $M_t(\theta)$ and applying Doob's inequality.

2 points for using $\lambda(\theta) \le \theta^2/2$ to obtain an exponential bound.

2 points for optimizing in θ and substituting $b = \sqrt{2n\log(1/\delta)}$.

Question 4. [9 points] Let p be a probability distribution on $\{0, 1, 2, \cdots\}$ satisfying $\mathbb{E}_{Z\sim p}[Z] = 1$. Consider a stochastic process $(X_n)_{n\geq 0}$ defined as follows: $X_0 = 1$ and for any $n \geq 1$, we have

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i},$$

where $Z_{n,i}$ are i.i.d. random variables following the distribution p and independent of X_{n-1}. If $X_n = 0$ for some n, we let $X_m = 0$ for all $m > n$, i.e., the process goes extinct once it hits zero. Assuming that p is not a degenerate distribution that puts all its mass on 1, show that

$$\mathbb{P}(\text{the process goes extinct}) = 1.$$

Answer: Let $\mathcal{F}_n = \sigma(Z_{k,i} : k \leq n, i \geq 1)$. Since the $Z_{n,i}$ are i.i.d. with mean $\mathbb{E}[Z] = 1$ and are independent of X_{n-1},

$$\mathbb{E}[X_n \mid \mathcal{F}_{n-1}] = \mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i} \,\middle|\, \mathcal{F}_{n-1}\right] = X_{n-1} \cdot \mathbb{E}[Z] = X_{n-1}.$$

Hence $(X_n)_{n\geq 0}$ is a martingale. Since $X_n \geq 0$ for all n and $\sup_n \mathbb{E}[X_n] = \mathbb{E}[X_0] = 1 < \infty$, it is a non-negative L^1-bounded martingale.

By the martingale convergence theorem, $X_n \to X_\infty$ a.s. for some non-negative random variable X_∞. Note that $X_n \in \{0, 1, 2, \ldots\}$ for every n. A sequence of integers can converge only if it is eventually constant, so $X_n = X_\infty$ for all sufficiently large n, almost surely.

For $k \geq 1$, if $X_n = k$, then the next value is $X_{n+1} = \sum_{i=1}^{k} Z_{n+1,i}$, which has mean k and variance $k\mathrm{Var}(Z) > 0$. So X_{n+1} is not almost surely equal to k. Therefore, k is not an absorbing state, and the process can escape from k with positive probability, and k cannot be the eventual constant value of X_n. Hence $\mathbb{P}(X_\infty = k) = 0$ for every $k \geq 1$. We can conclude that $X_\infty = 0$ almost surely, so the process goes extinct with probability one.

Rubrics: 4 points for identifying $(X_n)_{n\geq 0}$ as a non-negative martingale and applying the martingale convergence theorem to obtain an a.s. limit X_∞.

3 points for arguing that X_∞ is integer-valued and therefore eventually constant.

2 points for showing that the only possible eventual constant value is zero, hence the process goes extinct with probability one.

Question 5. [16 points] Sum of independent random variables

Part (a) [8 points]. Let $(X_n)_{n\geq 0}$ be i.i.d. normal random variables with mean zero and variance one. Define the process

$$S_n = \sum_{i=1}^{n} X_i^2.$$

Fix $a > 0$, define the stopping time

$$T := \inf\{n > 0 : |X_n| \geq a\}.$$

Compute $\mathbb{E}[S_T]$. You can express the answer using the cumulative distribution function of the standard normal distribution.

Answer: Let

$$p := \mathbb{P}(|X_1| \geq a) = 2\big(1 - \Phi(a)\big),$$

where Φ is the standard normal cdf. Since the X_n are i.i.d., the stopping time T is geometric with success probability p, so

$$\mathbb{E}[T] = \frac{1}{p} = \frac{1}{2(1-\Phi(a))}.$$

Now

$$S_T = \sum_{i=1}^{T} X_i^2,$$

and the random variables X_i^2 are i.i.d., nonnegative, with mean

$$\mathbb{E}[X_1^2] = 1.$$

Therefore Wald's identity applies and gives

$$\mathbb{E}[S_T] = \mathbb{E}[T]\mathbb{E}[X_1^2] = \frac{1}{2(1-\Phi(a))}.$$

Rubrics: 3 points for identifying T as a geometric random variable with parameter $2(1-\Phi(a))$.

5 points for correctly applying Wald's identity, or an equivalent argument, to obtain $\mathbb{E}[S_T] = 1/\{2(1-\Phi(a))\}$.

Part (b) [8 points]. Let $(Y_n)_{n\geq 0}$ be i.i.d. random variables with $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = 1/2$. Define the process

$$Z_n := \sum_{i=1}^{n} 2^{i-1} Y_i.$$

Define the stopping time

$$T := \inf\{n > 0 : Y_n > 0\}.$$

Compute $\mathbb{E}[T]$, and show that $\mathbb{E}[Z_T] \neq \mathbb{E}[Z_0]$.

Answer: Since T is the first time that $Y_n = 1$, it is geometric with parameter $1/2$. Hence

$$\mathbb{E}[T] = \frac{1}{1/2} = 2.$$

On the event $\{T = t\}$, we must have

$$Y_1 = Y_2 = \cdots = Y_{t-1} = -1, \qquad Y_t = 1.$$

Therefore

$$\begin{aligned} Z_T &= \sum_{i=1}^{t-1} 2^{i-1}(-1) + 2^{t-1} \\ &= -\sum_{i=1}^{t-1} 2^{i-1} + 2^{t-1} \\ &= -(2^{t-1} - 1) + 2^{t-1} = 1. \end{aligned}$$

So $Z_T = 1$ almost surely, and hence

$$\mathbb{E}[Z_T] = 1.$$

But $Z_0 = 0$, so

$$\mathbb{E}[Z_T] = 1 \neq 0 = \mathbb{E}[Z_0].$$

Rubrics: 3 points for correctly computing $\mathbb{E}[T] = 2$.
5 points for showing that $Z_T = 1$ almost surely, and therefore $\mathbb{E}[Z_T] \neq \mathbb{E}[Z_0]$.