STA447/2006 Lecture 1

wenlong.mou@utoronto.ca

A stochastic process $(X_t : t \in T)$.

For this course: interested in s.p. indexed by time.

Structures

- 4 weeks ← Markov chains (processes) (Discrete space)
  - $T = \mathbb{N}$ (Discrete-time MC)
  - $T = \mathbb{R}_+$ (cts-time discrete-space MC)
- 2–3 weeks ← Martingales — fair games/gambling
- 3–4 weeks ← Brownian motion / stochastic calculus. (cts-time, cts-space)

Def. (DTMC) Consider a state space S,

(Assume S finite / countably infinite)

- Initial distribution $(\nu_i)_{i \in S}$

  ($\nu_i \geq 0, \forall i, \quad \sum_{i \in S} \nu_i = 1$).

- Transition probabilities $P = (P_{ij})_{i,j \in S}$

  ($\forall i, \quad P_{i\cdot}$ is a probability distr. over S).

$(X_0, X_1, \cdots, X_n, \cdots)$ DTMC w/ (ν, P)

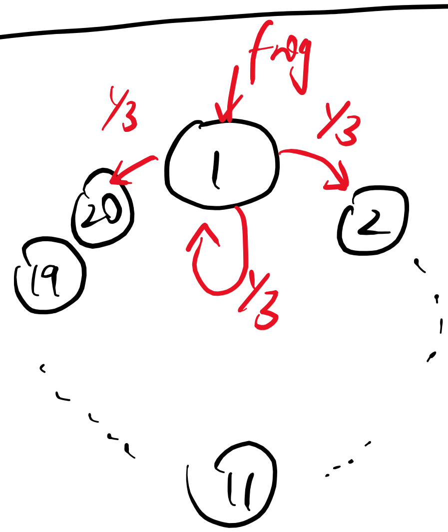
If, $\mathbb{P}(X_0 = i) = \nu_i$

$$\cdot\ \mathbb{P}(X_{t+1} = j \mid X_1 = x_1, X_2 = x_2, \cdots X_t = x_t)$$
$$= \mathbb{P}(X_{t+1} = j \mid X_t = x_t) = p(x_t, j).$$

Operationally, move at each step depends only on the current state

a.k.a. "Markov property"

e.g.



State space

$S = \{1, 2, \cdots, 20\}$

ν: initial state distribution

$$P_{ij} = \begin{cases} 1/3 & \text{when } i = j, \text{ or } i \equiv (j+1) \bmod 20. \\ & \text{or } i \equiv (j-1) \bmod 20. \\ 0 & \text{otherwise.} \end{cases}$$

$$P = \begin{bmatrix} 1/3 & 1/3 & 0 & \cdots & & 1/3 \\ 1/3 & 1/3 & 1/3 & & 0 & \vdots \\ 0 & & \ddots & \ddots & & 1/3 \\ 1/3 & \cdots & 0 & & 1/3 & 1/3 \end{bmatrix}$$

Throughout this class, we use

$$P = (P_{ij})_{i,j \in S}$$

row column.

$$\| \qquad \mathbb{P}(X_{t+1} = j \mid X_t = i).$$

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e.g. $S = \{0, 1, 2, \dots\}$. $\qquad X_0 = 0$ ($\nu_0 = 1$, $\nu_i = 0$ when $i > 0$).

At time t

$$X_{t+1} = X_t + \varepsilon_{t+1}$$

where $\varepsilon_t \overset{iid}{\sim} \mathrm{Ber}(1/2)$.

(Counting # heads in coin tosses).

$$P_{ij} = \begin{cases} 1/2 & i = j \\ 1/2 & i+1 = j \\ 0 & \text{otherwise.} \end{cases}$$

e.g. Ehrenfest's Urn. $\qquad$ d balls in total.

Box 1 $\qquad$ Box 2

At the step t.

— Randomly pick a ball (uniformly at random) within d balls

— Move it to the opposite side.

X_t = # balls in box 1 at time t.

$$S = \{0, 1, 2, \dots, d\}.$$

$$P_{ij} = \begin{cases} 1 - i/d & j = i+1 \\ i/d & j = i-1 \\ 0 & \text{otherwise.} \end{cases}$$

Non-example.

$$S = \mathbb{Z}. \qquad X_0 = 0.$$

$$X_{t+1} = X_t + \varepsilon_{t+1}$$

where given the history $(X_0, X_1, \dots, X_t)$.

$$\varepsilon_{t+1} \sim Ber(P_t)$$

and $p_t = \dfrac{\#\text{"up" moves within } [0,t] + 1}{t+2}$.

$(X_t)_{t\geq 0}$ is not a Markov chain.
(time inhomogeneous)

$(X_t, P_t)_{t\geq 0}$ is a Markov chain
(state space $\mathbb{Z} \times (\mathbb{Q} \cap (0,1))$).

Important properties of MCs.

"Markov property":

$$\mathbb{P}(X_t = j \mid X_1 = x_1, \dots X_{t-1} = x_{t-1}) = \mathbb{P}(X_t = j \mid X_{t-1} = x_{t-1})$$

Corollary: factorization of joint distribution.

$$\mathbb{P}(X_0 = x_0, X_1 = x_1, \dots X_n = x_n)$$
$$= \nu_{x_0} \cdot P_{x_0 x_1} \cdot P_{x_1 x_2} \cdots\cdots P_{x_{n-1}, x_n}.$$

(recursively applying Markov property).

e.g. $\mathbb{P}(X_2=j \mid X_0=i)$

$$= \sum_{k\in S} \mathbb{P}(X_2=j, X_1=k \mid X_0=i)$$

$$\overset{\text{(Markov)}}{=} \sum_{k\in S} \mathbb{P}(X_2=j \mid X_1=k)\cdot \mathbb{P}(X_1=k \mid X_0=i)$$

$$= \sum_{k\in S} P_{ik}\cdot P_{kj}.$$

when $|S|<+\infty$, this is matrix multiplication.

$$\mathbb{P}(X_2=j \mid X_0=i) = (P^2)_{i,j}$$

This is also true for countably infinite case.

In particular, if $A=(a_{ij})_{i,j\in S}$, $B=(b_{ij})_{i,j\in S}$ are transition matrices on countably infinite S.

$$[A\cdot B]_{i,j} = \sum_{k\in S} a_{ik} b_{kj}$$

We can show that $A\cdot B$ is also a transition matrix on S corresponding to the composition of the 2-step transitions.

In general, we have

$$\mathbb{P}(X_n = j \mid X_0 = i) = (P^n)_{i,j}$$

$$\mathbb{P}(X_n = j) = \sum_{i \in S} \nu_i \cdot (P^n)_{i,j} = [\nu \cdot P^n]_j$$

So the marginal distribution of X_n is $\nu \cdot P^n$

n-step transition matrix is P^n.

Notation:

$$p_{i,j}^{(n)} := [P^n]_{i,j} = \mathbb{P}(X_n = j \mid X_0 = i).$$

"Chapman-Kolmogorov equation":

$$p_{i,j}^{(m+n)} = (P^{m+n})_{i,j} = (P^m \cdot P^n)_{i,j}$$

$$= \sum_{k \in S} p_{ik}^{(m)} \cdot p_{kj}^{(n)}$$

"Strong Markov property".

Recall: Markov property.

$$X_0\ X_1 \cdots \Big|\ \underbrace{X_t, \cdots\ X_n \cdots}_{\text{fresh new process starting from } i.}$$

$$X_t = i.$$

What if we start from a random time?

e.g. $\tau = \inf\{t : X_t = i\}$

$$\underbrace{X_0 \cdots}_{\text{random length}}\ \Big|\ \underbrace{X_\tau\ X_{\tau+1} \cdots}_{\text{fresh new MC starting from } i.}$$

$$X_\tau = i$$

In general, strong Markov property holds for DTMC in discrete spaces, for "stopping times".

- hitting time of state i.
- k-th hitting time of state i.

"Long-time behavior" of MCs:

- convergence / divergence?
- Coming back (infinitely often)?
- Absorbing?

We need classification of states / MCs.

- Recurrence / transience.

Notation: for state $i \in S$.

$N(i) :=$ total # visits to i in MC trajectory (starting from $t=1$)

$$= \sum_{t=1}^{+\infty} \mathbb{1}_{X_t = i}.$$

(a random variable depending on the infinite MC trajectory).

$$f_{ij} := \mathbb{P}(N(j) \geq 1 \mid X_0 = i)$$

(prob. of ever visiting j if starting from i).

Def. A state $i \in S$ is called

$$\begin{cases} \text{recurrent} & \text{if } f_{ii} = 1 \\ \text{transient} & \text{if } f_{ii} < 1 \end{cases}$$

Additional notation. we denote by $i \to j$ the fact $f_{ij} > 0$.

($\exists$ path w/ positive prob. from i to j).

Fact: $$\mathbb{P}_i(N(i) \geq k) = f_{ii}^k$$

(shorthand notation for $\mathbb{P}(N(i) \geq k \mid X_0 = i)$)

If $f_{ii} = 1$, i is recurrent, and $N(i) = +\infty$ w.p. 1.

If $f_{ii} < 1$, $N(i) \sim \text{Geom}(f_{ii})$.

Proof of the fact: induction.

$k=1$. $\mathbb{P}_i(N(i)\geq 1) = f_{ii}$ by def.

Assuming that $\mathbb{P}_i(N(i)\geq k) = f_{ii}^k$ for certain $k\geq 0$.

For the case of $k+1$.

$$\mathbb{P}_i(N(i)\geq k+1)$$

$$= \underbrace{\mathbb{P}_i(N(i)\geq k)}_{= f_{ii}^k \text{ by induction hypothesis}} \cdot \mathbb{P}_i(N(i)\geq k+1 \mid N(i)\geq k).$$

Define $\tau_i^{(k)} :=$ k-th hitting time of i.

$$(\tau_i^{(k)} = \inf\{t > \tau_i^{(k-1)} : X_t = i\}).$$

$$\{N(i)\geq k\} = \{\tau_i^{(k)} < +\infty\}.$$

So we need to compute

$$\mathbb{P}(\tau_i^{(k+1)} < +\infty \mid \tau_i^{(k)} < +\infty).$$

$$\underset{\substack{\| \\ i}}{X_0}\quad X_1, \cdots\cdots, \underset{\substack{\| \\ i}}{X_{\tau_i^{(1)}}} \cdots\cdots \underset{\substack{\| \\ i}}{X_{\tau_i^{(2)}}} \cdots\cdots$$

$$\cdots\cdots \quad \boxed{\underset{\substack{\| \\ i}}{X_{\tau_i^{(k)}}} \cdots\cdots \quad \underset{\substack{\| \\ i}}{X_{\tau_i^{(k+1)}}}\ ?}$$

$(X_t)_{t \geq \tau_i^{(k)}}$ is a fresh new MC

starting from i.

(by strong Markov property).

So $$\mathbb{P}\left(\tau_i^{(k+1)} < +\infty \,\middle|\, X_0, X_1, \cdots\cdots X_{\tau_i^{(k)}}\right)$$

$$= \mathbb{P}_i\left(\tau_i^{(1)} < +\infty\right) = f_{ii}.$$

Substituting back completes the proof,

So for MC starting from i.

If $f_{ii} < 1$. then

$$\mathbb{P}_i(N(i) = k) = f_{ii}^k \cdot (1 - f_{ii}).$$

$$\mathbb{E}_i[N(i)] = \frac{f_{ii}}{1-f_{ii}}$$

(geometric distribution).

Furthermore, we can also compute

$$\mathbb{E}_i[N(j)] = \mathbb{P}_i(\text{visit } j) \cdot (1+\mathbb{E}_j[N(j)]).$$

fresh new MC.

$$\underset{\substack{\| \\ i}}{X_0} \;\cdots\cdots\; \underset{\substack{\| \\ j}}{X_\tau} \; \underbrace{X_{\tau+1} \cdots\cdots \cdots}_{N(j) \text{ visits to } j.}$$

$\tau =$ first visit time of $j (\neq i)$.

$$= f_{ij} \cdot \left(1+\frac{f_{jj}}{1-f_{jj}}\right)$$

$$= \frac{f_{ij}}{1-f_{jj}}. \qquad (\text{when } f_{jj}<1).$$

2 "corner cases":

- If $f_{ij}=0$. $\mathbb{E}_i[N(j)]=0$
- If $f_{ij}>0$, $f_{jj}=1$ then $\mathbb{E}_i[N(j)]=+\infty$.

Question: can we say sth. about recurrence/transience directly from P?

Thm (Recurrence State).

state $i \in S$ is recurrent if and only if

$$\sum_{n=1}^{+\infty} p_{ii}^{(n)} = +\infty.$$

. We can simply determine recurrence by computing the sum.

Application to concrete model.

. Simple random walk. $X_0 = 0$

$$X_{t+1} = X_t + \varepsilon_{t+1}$$

where $\varepsilon_t \overset{iid}{\sim} \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$

$$p_{i(i+1)} = p_{i(i-1)} = 1/2. \qquad p_{ij} = 0 \quad (\text{if } |i-j| \neq 1).$$

Question: is the state 0 recurrent?

$$\sum_{n=0}^{+\infty} p_{00}^{(n)} \overset{?}{=} +\infty.$$

$$P_{00}^{(n)} := \frac{\#\text{ paths that go back to } 0 \text{ at time } n}{2^n}$$

· If n is odd impossible.

· If n is even, # paths that get 0 at $t=n$

$$\|$$

up/down sequences with $n/2$ ups and $n/2$ downs.

$$\| $$

$$\binom{n}{n/2} = \frac{n!}{(n/2!)^2}$$

Stirling's approximation.

$$n! \approx \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

(to be rigorous, we can use Stirling's ineq to derive upper/lower bounds)

$$P_{00}^{(n)} = \frac{1}{2^n} \cdot \frac{n!}{(n/2)!^2} \approx \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\sqrt{n}}.$$

So we have

$$\sum_{n=1}^{+\infty} p_{00}^{(n)} = \text{const.} \sum_{n=1}^{+\infty} \frac{1}{\sqrt{n}} = +\infty.$$

So 0 is an recurrence state.

For 1-D SRW, you must be able to go back (though taking long time).

Multivariate SRW.

$$X_t \in \mathbb{Z}^d$$

$$X_{t+1} = X_t + \varepsilon_{t+1}.$$

where $\varepsilon_{t+1} \in \{-1, +1\}^d$

(each coordinate indp. ± 1 w.p. $1/2$)

$$p_{00}^{(n)} = \left(p_{00,\,1DSRW}^{(n)}\right)^d$$

$$= C_d \cdot n^{-d/2}.$$

when $d \leq 2$, $\sum_{n=1}^{+\infty} p_{00}^{(n)} = +\infty$

$d > 3$, $\sum_{n=1}^{+\infty} p_{00}^{(n)} < +\infty.$

A drunk man can get home
but a drunk bird may not