STA447/2006 Lecture 10

Final exam: April 22nd.

- Similar formats, requirements, etc
- 6 pages of cheat sheets (double-sided)
- Cover everything (uniformly).
- office hours (to be announced).

Recall: general formula.

$$dZ_t = X_t dt + Y_t dB_t.$$

Then.
$$df(t, Z_t) = \frac{\partial f}{\partial t}(t, Z_t)\,dt + \frac{\partial f}{\partial z}(t, Z_t)\,dZ_t + \frac{1}{2}\frac{\partial^2 f}{\partial z^2}(t, Z_t)\cdot Y_t^2\,dt$$

Corollary:

$$d\left(Z_t^{(1)} Z_t^{(2)}\right) = Z_t^{(1)} dZ_t^{(2)} + Z_t^{(2)} dZ_t^{(1)} + \underbrace{d\langle Z^{(1)}, Z^{(2)}\rangle_t}_{= Y_t^{(1)} Y_t^{(2)} dt}.$$

e.g. (Ornstein – Uhlenbeck process).

$$dZ_t = -Z_t\,dt + dB_t$$

($(Z_t)_{t\geq 0}$ is defined via stochastic differential eq.
Let's assume ∃! solution).

If we don't have dB_t, consider $(e^t Z_t)_{t\geq 0}$
is a constant

$$d(e^t Z_t) = e^t dZ_t + e^t Z_t\,dt + 0$$
$$= e^t dB_t.$$

$$Z_t = e^{-t}\Big(Z_0 + \underbrace{\int_0^t e^s\,dB_s}_{\text{Gaussian r.v. (by definition)}}\Big).$$

Marginally,

$$Z_t \sim N\left(e^{-t} Z_0,\; e^{-2t}\int_0^t e^{-2s}\,ds\right).$$

($(Z_t)_{t\geq 0}$ as a Markov process: exponential convergence to stationary).

eg. $dZ_t = rZ_t\,dt + bZ_t\,dB_t$, $Z_0 = 1$.

Guess the solution form

$$Z_t = \exp(at + kB_t).$$

$$dZ_t = a\cdot Z_t\,dt + k\cdot Z_t\,dB_t + \frac{k^2}{2} Z_t\,dt$$

Let $\begin{cases} k = b \\ a = r - b^2/2. \end{cases}$

So the solution is

$$Z_t = \exp\left(\left(r - \frac{b^2}{2}\right)t + bB_t\right).$$

. "geometric Brownian motion"

. When $r=0$. martingale.

Similar to the discrete-time multiplicative gambling.

Multivariate versions.

$$B_t = \left(B_t^{(1)}, B_t^{(2)}, \cdots, B_t^{(d)}\right) \in \mathbb{R}^d,$$

df each $B_t^{(i)}$ are independent.

$$df(t, B_t) = \frac{\partial f}{\partial t}(t, B_t)\,dt + \nabla_x f(t, B_t)^T dB_t + \frac{1}{2}\Delta_x f(t, B_t)\,dt$$

$$\left(\Delta_x f(t,x) = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}(t,x)\right)$$

Generally,

$$dZ_t = X_t\,dt + Y_t\,dB_t \qquad \left(X_t \in \mathbb{R}^d,\ Y_t \in \mathbb{R}^{d\times d}\right)$$

$$df(t, Z_t) = \frac{\partial f}{\partial t}(t, Z_t)\,dt + \nabla_x f(t, Z_t)^T dZ_t + \frac{1}{2}\mathrm{Tr}\left(Y_t^T \nabla_x^2 f(t, Z_t)\, Y_t\right)dt.$$

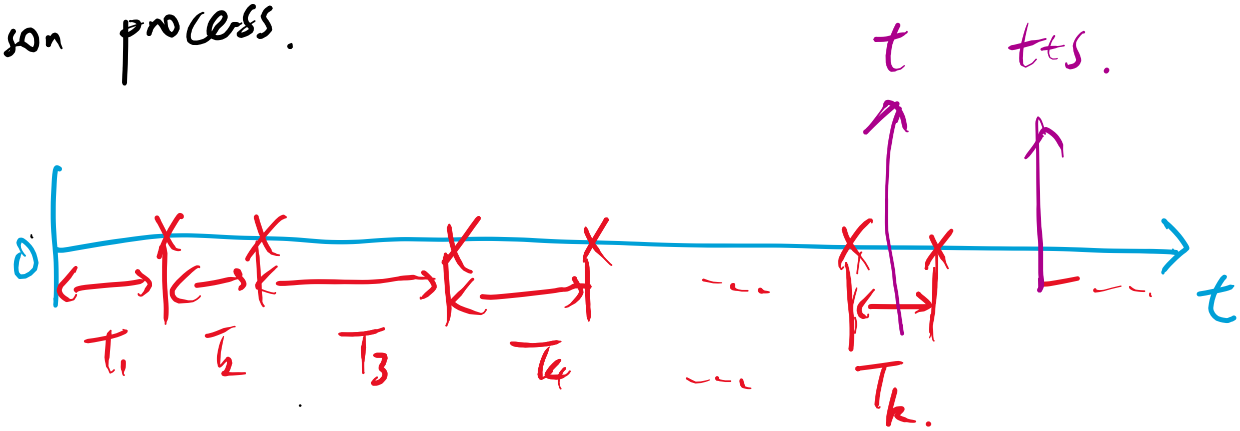
Continuous-time Discrete-space processes.

- Poisson process review.

$$N \sim \mathrm{Poi}(\lambda) \quad \text{if} \quad \mathbb{P}(N=n) = \frac{e^{-\lambda}\lambda^n}{n!}$$

$$(n = 0, 1, 2, \cdots)$$

Poisson process.



$T_i \overset{iid}{\sim} \text{Exp}(\lambda)$. $\quad p_\lambda(t) = \begin{cases} \lambda e^{-\lambda t} & (t \geq 0) \\ 0 & (\text{otherwise}) \end{cases}$

$N(t) :=$ # marked pts within $[0, t]$

Why exp time? — Markov property.

$$\mathbb{P}(T_i \geq t+s \mid T_i \geq t) = \mathbb{P}(T_i \geq s).$$

Facts.

- Marginally. $N(t) \sim \text{Poi}(\lambda t)$
- Markov.
- Indp Poisson increments.

$$\underbrace{N(t+s) - N(t) \sim \text{Poi}(\lambda s)}$$
indp of the past.

- $(N(t)-\lambda t)_{t\geq 0}$ is MG.

$$\mathbb{E}[N(t)-\lambda t \mid \mathcal{F}_s] = \mathbb{E}[N(t)-N(s) \mid \mathcal{F}_s] + (N(s)-\lambda s) - \lambda(t-s)$$
$$= N(s) - \lambda s.$$

- Axiomic definition, non-constructive.

$(N(t))_{t\geq 0}$ taking non-neg int values, non-dec.

- $\mathbb{P}(N(t+h)-N(t)=1) = \lambda h + o(h)$ $(h \to 0)$.
- $\mathbb{P}(N(t+h)-N(t)\geq 2) = o(h)$.

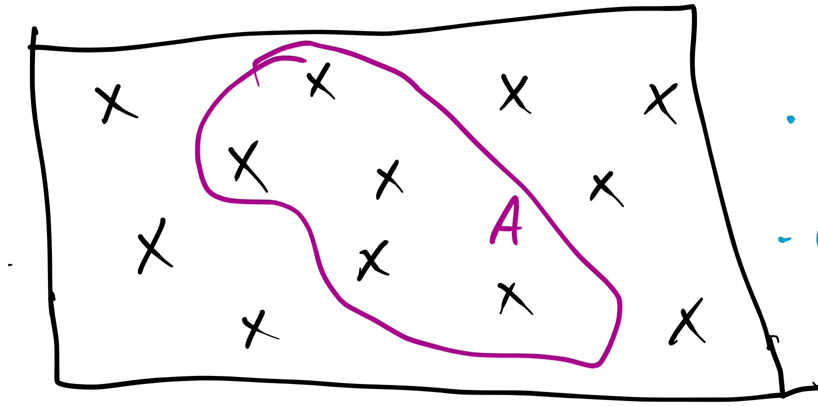
(PP(λ) is the only process satisfying this)

Generalization:

$$\mathbb{P}(N(t+h)-N(t)=1 \mid \mathcal{F}_t) = \lambda(t, N(t))h + o(h)$$
$$\mathbb{P}(N(t+h)-N(t)\geq 2 \mid \mathcal{F}_t) = o(h).$$

Markov process. SDE driven by PP instead of BM

- Extension: Poisson point process.



Construction:

- Sample $N \sim \text{Poi}\left(\int_S \lambda(x)\,dx\right)$.
- Conditioned on N, sample N i.i.d. marked pts $p(x) = \dfrac{\lambda(x)}{\int_S \lambda(y)\,dy}$

$N(A) := \#$ marked pts in A for $A \subseteq S$.

PPP: • $N(A) \sim \text{Poi}\left(\int_A \lambda(x)\,dx\right)$.

• For disjoint A, B, $N(A) \perp N(B)$.

- $PP(\lambda)$ is a special case of PPP with $S = \mathbb{R}^+$, $\lambda(x) \equiv \lambda$.

- Construction specialized to PP: equivalent def.

  — Sample $N(T) \sim \text{Poi}(\lambda T)$

  — Conditioned on N, sample N iid marked pts $\sim \text{Unif}([0, T])$.

— Let $N(t) := \#$ marked pts within $[0,t]$
$(\forall t \in [0,T])$

From this construction, we can derive:

- Superposition property:

Let $(N_1(t))_{t\geq 0}$ and $(N_2(t))_{t\geq 0}$ indp PP

with intensity λ_1, λ_2.

Then $\left(N_1(t)+N_2(t)\right)_{t\geq 0} \sim PP(\lambda_1+\lambda_2)$

- "Thinning property":

$(N(t))_{t\geq 0} \sim PP(\lambda)$.

For each marked pt, label it with type i

w.p. p_i, iid across marked pts.

$\forall i$, $N_i(t) := \#$ marked pts of type i in $[0,t]$

$(N_i(t))_{i\geq 0}$ are indp $PP(\lambda p_i)$

- Cts-time Discrete-space Markov processes.

Generally, consider state space S (finite or countable)

initial distribution ν on S.

Defn. $(X(t))_{t\geq 0}$ Markov if

$$\forall\ 0<t_1<t_2<\cdots<t_n.$$

$$\mathbb{P}(X_0=i_0, X_1=i_1, \cdots X_{t_n}=i_n)$$

$$=\nu_{i_0}\, P^{(t_1)}_{i_0,i_1}\, P^{(t_2-t_1)}_{i_1,i_2} \cdots P^{(t_n-t_{n-1})}_{i_{n-1},i_n}$$

- eg. $PP(\lambda)$ is a CTMC.

$$P^{(t)}_{ij} = \begin{cases} 0 & (j<i) \\ \dfrac{e^{-\lambda t}(\lambda t)^{j-i}}{(j-i)!} & (j\geq i). \end{cases}$$

Def.

$$g_{ij} = \lim_{t\to 0^+} \frac{P^{(t)}_{ij}-\delta_{ij}}{t} = \frac{d}{dt}P^{(t)}_{ij}\Big|_{t=0}.$$

$$\left(\text{where } \delta_{ij} = \mathbb{1}_{\{i=j\}}.\right)$$

under mild conditions,

generator $(g_{ij})_{i,j\in S}$ exists.

Basic properties:

- $g_{ii} \le 0 \quad (\forall i \in S)$.
- $g_{ij} \ge 0 \quad (i \ne j)$.
- $\sum_{j\in S} g_{ij} = \lim_{t\to 0} \frac{\sum_{j\in S} p_{ij}^{(t)} - 1}{t} = 0.$

Thm. If G is generator,

then $P^{(t)} = (p_{ij}^{(t)})_{i,j\in S} = \exp(tG)$.

$$\left(\exp(tG) = I + tG + \frac{t^2G^2}{2!} + \cdots + \frac{t^nG^n}{n!} + \cdots \right)$$

(Computationally, you can exponentiate eigen-values).

e.g. $PP(\lambda)$

$$G = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots & & 0 \\ 0 & -\lambda & \lambda & 0 & \cdots & \cdots \\ 0 & 0 & -\lambda & \lambda & 0 & \cdots \\ & & & \ddots & \ddots & 0 \\ & & 0 & & \ddots & \ddots \end{bmatrix}$$

Proof idea:

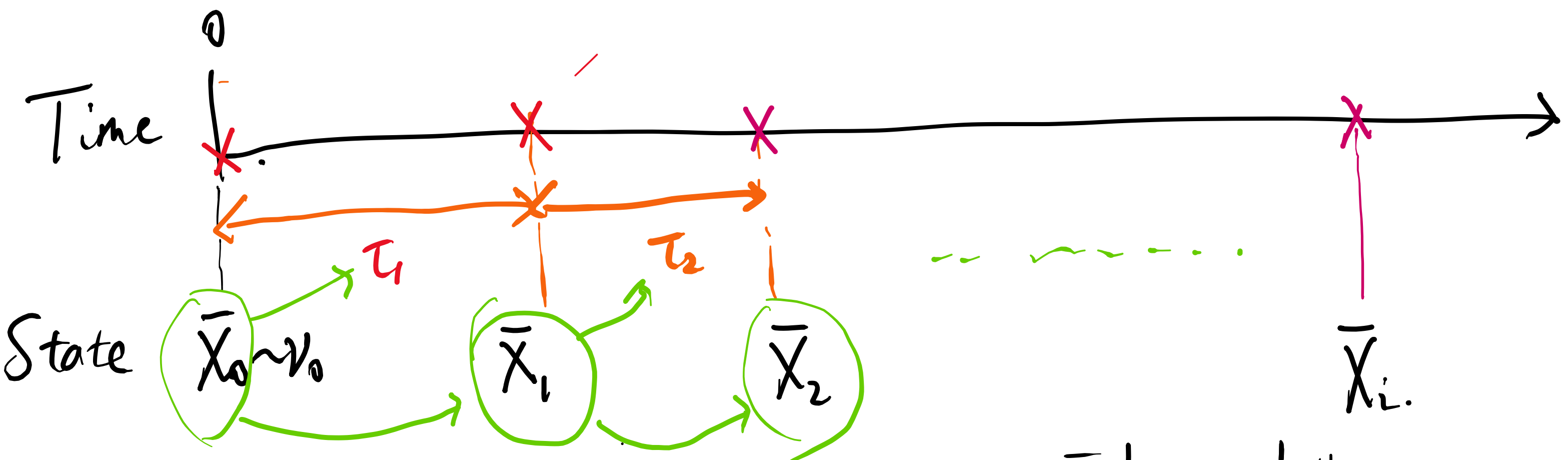
$$P^{(t)} = \left(P^{(t/n)}\right)^n \quad \text{take } n \to +\infty.$$

Taylor expansion, $P^{(t/n)} = I + \frac{t}{n} G + o\left(\frac{1}{n}\right)$.

Constructive characterization.

Given generator $G = (g_{ij})_{i,j \in S}$.

How does $(X_t)_{t \geq 0}$ look like?



$$X_t = \bar{X}_k \quad \text{when } t \in \left[\sum_{i=1}^{k} \tau_i, \sum_{i=1}^{k+1} \tau_i\right).$$

Conditionally on first i jumps

at time $\sum_{j=1}^{i} \tau_j$

Based on $\bar{X}_i$, we independently sample

$$\begin{cases} \tau_{i+1} \sim \mathrm{Exp}(|g_{xx}|) \quad \text{where } x = \bar{X}_i \\ \quad \text{(if } g_{xx}=0, \text{ absorbing state).} \\ \bar{X}_{i+1} \sim \left(\dfrac{g_{x,y}}{|g_{xx}|} \right)_{y \in S}. \end{cases}$$

All DTMC properties transfer to here.

- Recurrence/transience
- Convergence to stat.
- Time averaging — weight by $|g_{xx}|$.

Final review.

- DTMC.
  - Recurrence/transience
  - f-expansion, hitting time computation.

- Convergence to stationary, positive recurrence.

- DT martingales.
  - OST
  - Convergence
  - Conditions for these theorems. (uniform integrability).

- CT processes.
  - BM. reflection principle.
  - Stochastic calculus.
  - PP and CTMC, equivalent characterizations.