STA447/2006 Lecture 2

Recall from last time.

Thm (recurrence state thm).

State i is recurrent $\Longleftrightarrow$ $\sum_{n=1}^{+\infty} P_{ii}^{(n)} = +\infty.$

(w.p. 1, MC starting from i will return to i infinitely often)

$P_{ii}^{(n)}$ = n-step transition.

e.g. application, multidimensional RW.

$$\sum_{n=1}^{+\infty} P_{ii}^{(n)} = \sum_{n=1}^{+\infty} \left(\frac{c}{\sqrt{n}}\right)^d \begin{cases} < +\infty & d \geq 3 \\ = +\infty & d = 1, 2. \end{cases}$$

Proof of the theorem.

Detour: Borel-Cantelli Lemma.

General tool for reasoning infinite seq of events (finitely or infinitely often)

Let $(E_n)_{n=1}^{+\infty}$ be a sequence of events.

If $\sum_{n=1}^{+\infty} \mathbb{P}(E_n) < +\infty$

then

$$\mathbb{P}\left((E_n)_{n=1}^{+\infty} \text{ happens only finitely often}\right) = 1.$$

Note: converse is false in general.

Application to recurrence state thm:

We let $E_n = \{X_n = i\}$ (visit back i at time n).

If $\sum_{n=1}^{+\infty} \mathbb{P}(E_n) = \sum_{n=1}^{+\infty} p_{ii}^{(n)} < +\infty$

then by BC lemma. $(E_n)_{n=1}^{+\infty}$ happens only finitely many times, w.p. 1

i.e., we only visit back to i finitely many times, w.p. 1.

This implies transience. and proves "$\Rightarrow$" of thm.

Back to the detour.

Proof of BC lemma.

$$\sum_{n=1}^{+\infty} \mathbb{P}(E_n) \underset{\substack{\text{Fubini}\\ \text{-Tonelli}}}{=} \mathbb{E}\left[\sum_{n=1}^{+\infty} \mathbb{1}_{E_n}\right] < +\infty \quad \text{(by assumption)}.$$

$\uparrow$

\# of times event happens.

So the r.v. $\sum_{n=1}^{+\infty} \mathbb{1}_{E_n}$ has finite expectation.

and thus finite a.s.

From the proof, it's clear that converse cannot be guaranteed —— $\exists$ real-valued r.v. w/ infinite $\mathbb{E}[\cdots]$.

However, for MC's. $N(i) = \sum_{n=1}^{+\infty} \mathbb{1}\{X_n = i\}$

is either infinite w.p. 1 or

finite & geometrically distributed.

Using this observation, we can prove "$\Leftarrow$" of recurrence state thm.

$$\sum_{n=1}^{+\infty} p_{ii}^{(n)} \underset{\text{(Fubini-Tonelli)}}{=} \mathbb{E}_i\left[\sum_{n=1}^{+\infty} \mathbb{1}_{X_n = i}\right]$$

$$= \mathbb{E}_i[N(i)].$$

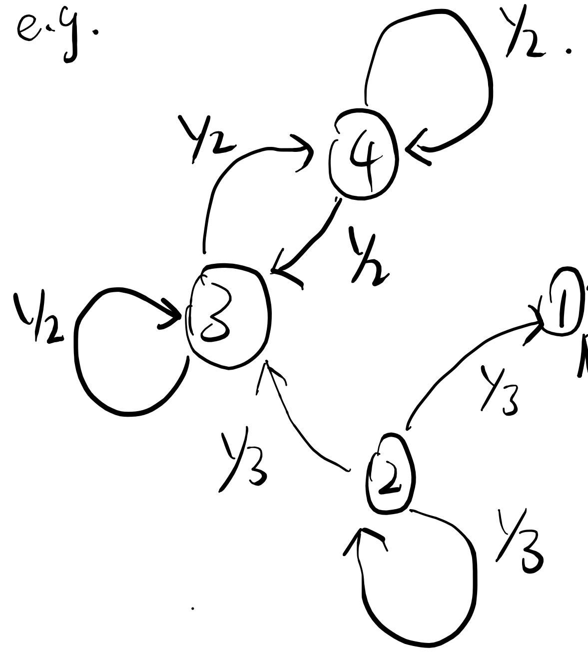
If i is transient, then $f_{ii} < 1$

$$\mathbb{E}_i[N(i)] = \frac{f_{ii}}{1 - f_{ii}} < +\infty.$$

This proves "$\Leftarrow$" of thm.

Next question: how to compute f_{ij}'s systematically?

Motivating e.g.



$$f_{11} = 1. \quad f_{1j} = 0 \quad \text{for } j \in \{2,3,4\}.$$

$$f_{22} = \frac{1}{3}.$$

$$f_{21} = \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} + \left(\frac{1}{3}\right)^3 + \cdots$$

$$= \frac{1/3}{1 - 1/3} = \frac{1}{2}.$$

Similarly. $f_{23} = f_{24} = \frac{1}{2}$

In general, we use f-expansion.

i.e. solving out f_{ij}'s using linear systems.

$$f_{ij} = \mathbb{P}_i(\text{visit } j).$$

(Expand based on the first step)

$$= \sum_{k\in S} \mathbb{P}_i(X_1 = k)\cdot \mathbb{P}_k(\text{visit } j \text{ starting from time } 0).$$

$$\left(\neq \sum_{k\in S} p_{ik} f_{kj}\right)$$

$$= \sum_{\substack{k\in S \\ k\neq j}} p_{ik} f_{kj} + p_{ij}$$

If $|S| < +\infty$, f-expansion gives $|S|^2$ equations for $|S|^2$ variables.

We can solve out these variables.

A more interesting example.

— Gambler's ruin.

Suppose a gambler plays coin-tossing games.

- Initially, $a \in \mathbb{N}_+$ units of money.
- For each round, bet 1 unit

with winning prob $p \in (0,1)$.

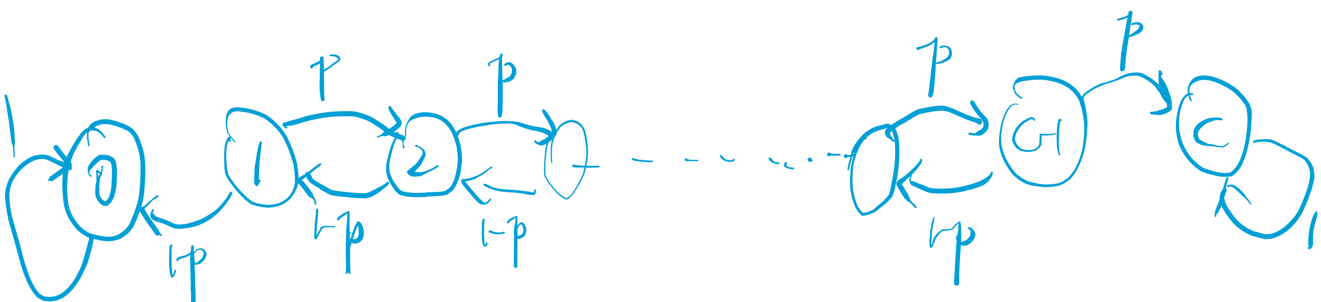
- Stop when $\begin{cases} \text{money} = 0 \\ \text{money} = c > a. \end{cases}$

In MC language, let X_n be the amount of money in n-th round.

$$X_0 = a. \qquad X_{n+1} = X_n + \varepsilon_{n+1} \text{ when } X_n \in \{1, 2, \dots, c-1\}$$

$$\text{where } \varepsilon_{n+1} = \begin{cases} +1 & \text{w.p. } p \\ -1 & \text{w.p. } (1-p) \end{cases} \quad \text{indp.}$$

$$X_{n+1} = X_n \quad \text{when } X_n \in \{0, c\}.$$



Interested in ruin probability

$$\mathbb{P}_a(\text{visit } 0) = f_{a0}.$$

Computation of f_{a0}:

$$f_{00} = 1 \qquad f_{c0} = 0$$

By f-expansion.

$$f_{a0} = p f_{(a+1)0} + (1-p) f_{(a-1)0}$$

(when $a \geq 2$)

For $a=1$

$$f_{10} = p f_{20} + (1-p)$$

$$\left(= p f_{20} + (1-p) f_{00} \quad \text{since } f_{00} = 1 \right)$$

So $\quad f_{a0} = p f_{(a+1)0} + (1-p) f_{(a-1)0} \quad$ for $a \in \{1, 2, \dots c-1\}$

Solving the f-expansion eqs.

Case I: $p = 1/2$.

$$f_{a0} = \frac{1}{2}\left(f_{(a+1)0} + f_{(a-1)0} \right),$$

So $(f_{a0})_{a=0}^{c}$ is an arithmetic sequence, i.e.

$$f_{c0} - f_{(c-1)0} = f_{(a+1)0} - f_{a0} = f_{a0} - f_{(a-1)0} = \cdots = f_{10} - f_{00}$$

$$\| \qquad\qquad -\frac{1}{c}$$

$f_{00} = 1, \ f_{c0} = 0$

and therefore,

$$f_{a0} = 1 - \frac{a}{c}.$$

Similarly, $$f_{ac} = \frac{a}{c}.$$

Remark: We cannot make money from it (in expectation).

$$\mathbb{E}[X_{final}] = c \cdot \frac{a}{c} + 0 \cdot \left(1 - \frac{a}{c}\right) = a.$$

(Indeed, this is not a coincidence) c.f. martingale part.

Case II: $p \neq 1/2$.

$$f_{a0} = p f_{(a+1)0} + (1-p) f_{(a-1)0}$$

$$p f_{(a+1)0} - p f_{a0} + (1-p) f_{(a-1)0} - (1-p) f_{a0} = 0.$$

So

$$f_{(a+1)0} - f_{a0} = \frac{1-p}{p}\left(f_{a0} - f_{(a-1)0}\right).$$

The increments form a geometric seq.

$$f_{(a+1)0} - f_{a0} = \left(\frac{1-p}{p}\right)^a \left(f_{10} - f_{00}\right).$$

Summing up over a,

$$-1 = f_{c0} - f_{00} = \sum_{a=0}^{c-1} \left(f_{(a+1)0} - f_{a0}\right)$$

$$= \left(f_{10} - f_{00}\right) \cdot \left[\sum_{a=0}^{c-1} \left(\frac{1-p}{p}\right)^a\right].$$

Following some calculation,

$$f_{a0} = \frac{\left(\frac{1-p}{p}\right)^c - \left(\frac{1-p}{p}\right)^a}{\left(\frac{1-p}{p}\right)^c - 1} \qquad \left(p \neq \tfrac{1}{2}\right).$$

Remark: when $p \neq \frac{1}{2}$.

ruin prob is $\begin{cases} \text{exponentially small when } p > \frac{1}{2}. \\ \text{exponentially close to 1 when } p < \frac{1}{2}. \end{cases}$

Relation between recurrence/transience of different states.

Recall notation:

$i \to j$ means j is reachable from i, $f_{ij} > 0$.

$i \leftrightarrow j$ means $i \to j$ and $j \to i$.

Def. We call an MC irreducible when $i \leftrightarrow j$ for $i, j \in S$.

(Implicitly, an MC not satisfying this can be "reduced" into smaller ones).

Fact. If $i \leftrightarrow j$, then "i is recurrent"

$$\Updownarrow$$

"j is recurrent".

Corollary of the fact: "case theorem"

If a MC is irreducible, then one of the following is true:

- All states are recurrent. ("recurrent MC")
- All states are transient ("trancient MC").

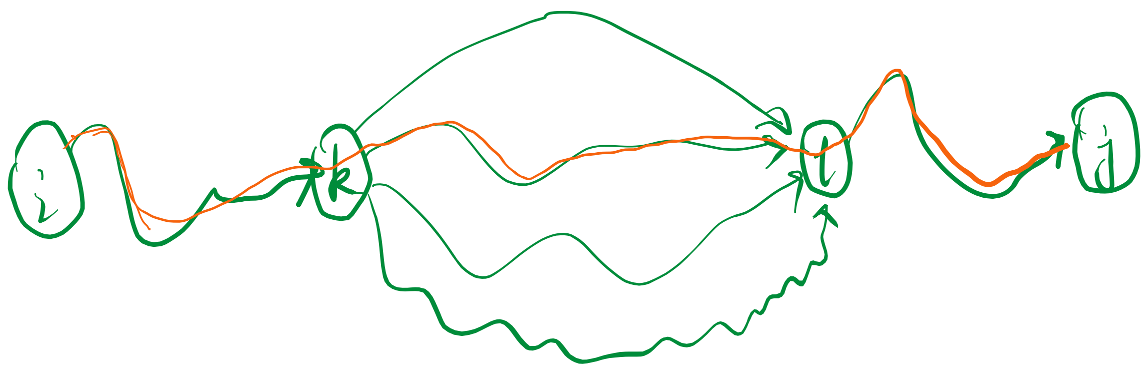
Proof of the fact:

We use the "sum lemma".

Lemma: If $i, j, k, l \in S$ (allow repeating).

Suppose $i \to k$ and $l \to j$

If $\sum_{n=1}^{+\infty} p_{kl}^{(n)} = +\infty$ then $\sum_{n=1}^{+\infty} p_{ij}^{(n)} = +\infty$.



Proof: Since $i \to k$, $\exists\, m > 0$, $p_{ik}^{(m)} > 0$

$l \to j$, $\exists\, r > 0$ $p_{lj}^{(r)} > 0$.

Then (by Kolmogorov – Chapman equation)

$$p_{ij}^{(n+m+r)} \geqslant p_{ik}^{(m)} \cdot p_{kl}^{(n)} \cdot p_{lj}^{(r)}.$$

um them together

$$\sum_{n=1}^{+\infty} p_{ij}^{(n)} \geq \underbrace{p_{ik}^{(m)}}_{>0} \cdot \underbrace{p_{lj}^{(r)}}_{>0} \cdot \underbrace{\sum_{n=1}^{+\infty} p_{kl}^{(n)}}_{=+\infty} = +\infty.$$

This ves sum lemma

I ch to prove of recurrence equivalence:

Apply sum lemma w/ $j=i$, $l=k$.

$$i \to k,\ l \to j \iff i \leftrightarrow l.$$

l is recurrent $\iff \sum_{n=1}^{+\infty} p_{ll}^{(n)} = +\infty.$

sum lemma implies $\sum_{n=1}^{+\infty} p_{ii}^{(n)} = +\infty.$

which proves desired fact.

Next question.

What do we know about $\sum_{n=1}^{+\infty} p_{ij}^{(n)}$? (for $i \neq j$).

Fact. If MC is $\begin{cases} \text{Recurrent}, & \sum_{n=1}^{+\infty} p_{ij}^{(n)} = +\infty \\ \text{Transient}, & \sum_{n=1}^{+\infty} p_{ij}^{(n)} < +\infty. \end{cases}$

(This is if-and-only-if: any i,j pair is equivalent to MC recurrence/transience)

Proof. $\sum_{n=1}^{+\infty} p_{ij}^{(n)} = \mathbb{E}_i[N(j)]$.

- Recurrent, $\exists m,\ p_{ij}^{(m)} > 0$ by irreducibility

$$\mathbb{E}_i[N(j)] \geq p_{ij}^{(m)} \cdot \underbrace{\mathbb{E}_j[N(j)]}_{=+\infty \text{ by recurrence}} = +\infty.$$

- Transient,

$$\mathbb{E}_i[N(j)] = f_{ij} \cdot (\underbrace{\mathbb{E}_j[N(j)]}_{=\frac{f_{jj}}{1-f_{jj}}} + 1)$$

$$= \frac{f_{ij}}{1-f_{jj}} < +\infty.$$

e.g. When $|S| < +\infty$, and irreducible, then the MC is recurrent

Proof. For any fixed $i \in S$.

$$\underbrace{\sum_{j \in S}\left(\sum_{n=1}^{+\infty} p_{ij}^{(n)}\right)}_{\text{sum of finitely many terms.}} = \sum_{n=1}^{+\infty}\left(\sum_{j \in S} p_{ij}^{(n)}\right) = \sum_{n=1}^{+\infty} 1 = +\infty.$$

$\exists\, j_0$, s.t.

$$\sum_{n=1}^{+\infty} P_{ij_0}^{(n)} = +\infty.$$

and by irreducibility, this implies MC is recurrent.

Another useful fact: "f-lemma".

Motivation: $\exists$ characterization for recurrence / transience

$$\begin{cases} \sum_{n=1}^{+\infty} P_{ij}^{(n)} & \text{equivalent} \\ N(j) & \text{the same} \\ f_{ij} \ ? & \end{cases}$$

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f Lemma. If $j \to i$, $f_{jj} = 1$

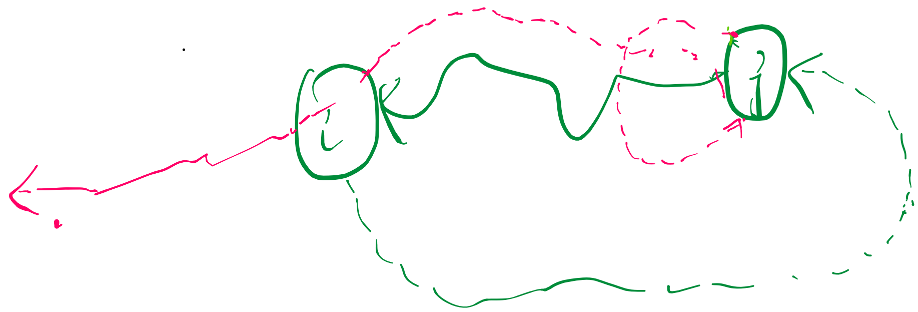
then $f_{ij} = 1$.

(Compute off-diagonal f for recurrent states).

Direct corollary:

For recurrent MC, $f_{ij} = 1$ for $i, j \in S$.

Proof of f-Lemma.



Technical tool: "hit lemma".

Define the event

$$H_{ij} := \{\text{MC hits } i \text{ before returning to } j\}.$$

f $j \to i$, then $\mathbb{P}_j(H_{ij}) > 0$.

(This lemma removes the trouble of "$\dashrightarrow$" path).

Proof of "Hit lemma".

$j \to i$. So $\exists\ x_0\ x_1 \cdots\ x_\ell \cdots x_{m-1}\ x_m$, with $x_0 = j$, $x_\ell = j$, $x_m = i$.

We have

$$p_{x_0 x_1}\, p_{x_1 x_2} \cdots p_{x_{m-1} x_m} > 0.$$

This implies

$$p_{x_\ell x_{\ell+1}} p_{x_{\ell+1} x_{\ell+2}} \cdots p_{x_{m-1} x_m} > 0.$$

Let ℓ be the last time j appears in this sequence.

$$\mathbb{P}_j(H_{ij}) \geq p_{x_\ell x_{\ell+1}} \cdots p_{x_{m-1} x_m} > 0$$

which proves hit lemma.

Proof of f-lemma using hit lemma.

$$0 = \mathbb{P}_j(\text{never return to } j)$$
$$\geq \underbrace{\mathbb{P}_j(H_{ij})}_{>0} \cdot \underbrace{\boxed{\mathbb{P}_i(\text{never visit } j)}}_{\text{must be } 0.}$$

So $f_{ij} = 1$, which proves f-lemma.