STA447/2006 Lecture 3

Recall : irreducible M.C.

$\exists i \in S$ recurrent / transient $\iff$ $\forall i \in S$ r/t.

Equivalent cond. for recurrent chains.

$\boxed{\forall \text{ or } \exists\ i,j \in S,\ \sum_{n=1}^{+\infty} P_{ij}^{(n)} = +\infty}$

$\Updownarrow$ (sum lemma)

$\boxed{\forall \text{ or } \exists\ i \in S \quad f_{ii} = 1} \iff$ Recurrence $\iff \boxed{\forall \text{ or } \exists\ i,j \in S \quad \mathbb{P}_i(N(j) = +\infty) = 1}$

$\Updownarrow$ ($\forall$ -lemma)

$\boxed{\forall\ i,j \in S \quad f_{ij} = 1}$

$\boxed{\exists\ i,j \in S,\ f_{ij} = 1}$

$\boxed{\forall \text{ or } \exists\ i,j \in S \quad f_{ij} = f_{ji} = 1}$

"Infinite returns lemma"

Recurrent $\Rightarrow \forall i,j \in S \quad \mathbb{P}_i(N(j) = +\infty) = 1$

Transient $\Rightarrow \forall i,j \in S \quad \mathbb{P}_i(N(j) = +\infty) = 0.$

Proof. For recurrent case, by f-lemma.

$f_{ij} = 1$, starting from j

$\mathbb{P}_j(N(j) = +\infty) = 1.$

For transient case.

$$\mathbb{E}_i[N(j)] = \frac{f_{ij}}{1 - f_{jj}} < +\infty.$$

Counterexample of "$\not\to$", we'll show

$\exists$ an irreducible MC s.t. $\exists\, i,j \in S$ $f_{ij} = 1$

but transient.

Construction: $X_0 = 0$

$$X_{n+1} = X_n + \varepsilon_{n+1}, \quad \varepsilon_{n+1} \overset{iid}{\sim} \begin{cases} 1 & \text{w.p. } 2/3 \\ -1 & \text{w.p. } 1/3. \end{cases}$$

- Irreducible. ✓
- Transient.

$$P_{00}^{(n)} = \begin{cases} 0 & (n \text{ is odd}). \\ \binom{n}{n/2} \cdot \left(\frac{1}{3}\right)^{n/2} \cdot \left(\frac{2}{3}\right)^{n/2} & (n \text{ is even}). \end{cases}$$

$$\left(\text{by stirling approx.} \approx \frac{c}{\sqrt{n}} \left(\frac{8}{9}\right)^{n/2} \right).$$

$$\sum_{n=1}^{+\infty} P_{00}^{(n)} < +\infty \Rightarrow \text{transience.}$$

- Claim: $f_{01} = 1$. (but $f_{10} < 1$).

Proof. By SLLN, $\frac{X_n}{n} \xrightarrow{a.s.} \mathbb{E}[\varepsilon_i] = \frac{1}{3} > 0.$

So $X_n \to +\infty$ w.p. 1

On this prob. 1. event, $(X_n)_{n\geq 1}$ must visit 1.

So $f_{01} = 1$.

On the other hand, if we assume

$$f_{ij} = f_{ji} = 1 \quad \text{for some } i,j \in S$$

this implies $f_{ii} = 1$ and therefore recurrence.

Similarly, transience equivalence thm.

$$\exists \text{ or } \forall\ i,j \in S,\ \sum_{n\geq 1}^{+\infty} P_{ij}^{(n)} < +\infty$$

$$\Updownarrow$$

$$\boxed{\forall \text{ or } \exists\, i \in S \quad f_{ii} < 1.} \iff \text{Transience} \iff \exists \text{ or } \forall\ i,j \in S \quad \mathbb{P}_i(N(j) < +\infty) = 1$$

$$\Updownarrow$$

$$\boxed{\exists\ i,j \in S \quad f_{ij} < 1}$$

$$\text{Transience} \not\Longrightarrow \forall\ i,j \in S,\ f_{ij} < 1.$$

Reducible MC.

For finite state space, structurally like gambler's ruin.

$$P = \begin{array}{c}\left[\begin{array}{cccc|c} P_1 & 0 & \cdots & 0 & \\ 0 & P_2 & & 0 & \\ 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & \cdots\, 0 & P_K & \\ \hline S_1 & S_2 & \cdots & S_K & Q \end{array}\right]\end{array} \begin{array}{l} \left.\begin{array}{c}\\ \\ \\ \\ \end{array}\right\} \text{Recurrent states} \\ \left.\begin{array}{c}\\ \end{array}\right\} \text{Transient states}\end{array}$$

Impossible to go from recurrent state to transient state (f-Lemma)

If $f_{ii} = 1$, $f_{ij} > 0$, then j is recurrent.

Next question:

$$\lim_{n\to+\infty} \mathbb{P}_i(X_n = j) \quad \text{or} \quad \lim_{n\to+\infty} \frac{1}{n}\sum_{t=1}^{n} \mathbb{P}_i(X_t = j). \ ?$$

Suppose the convergence happens

$$p_{ij}^{(n)} \to q_j \quad (\forall i, j \in S)$$

then $$P_{ij}^{(n+1)} = \sum_{k\in S} P_{ik}^{(n)} \cdot P_{kj}$$

Taking limit on both sides

$$q_j = \sum_{k\in S} q_k P_{kj}. \quad (\forall j \in S)$$

i.e. $q = qP.$

Def. We say π is a stationary distribution of MC P when

- π is a prob. distr.
- $\pi = \pi P$ (thinking π as a row vector).

$$\pi_j = \sum_{i\in S} \pi_i P_{ij} \quad \forall j \in S.$$

Def. (relaxing the first condition).

π is a stationary measure for P.

- $\pi(i) \geq 0$ $(\forall i)$, $\pi \neq 0$. $\sum_{i\in S} \pi(i)$ can be infinite.
- $\pi = \pi P.$

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e.g. Frog walk on 20 states

$$\pi = \left[\frac{1}{20}, \frac{1}{20}, \cdots, \frac{1}{20}\right]$$

Easy to verify

$$\frac{1}{20} = \pi_i \underset{\checkmark}{\overset{?}{=}} \sum_{j\in S} \pi_j P_{ji} = \frac{1}{20}\times\frac{1}{3}\times 3 = \frac{1}{20}.$$

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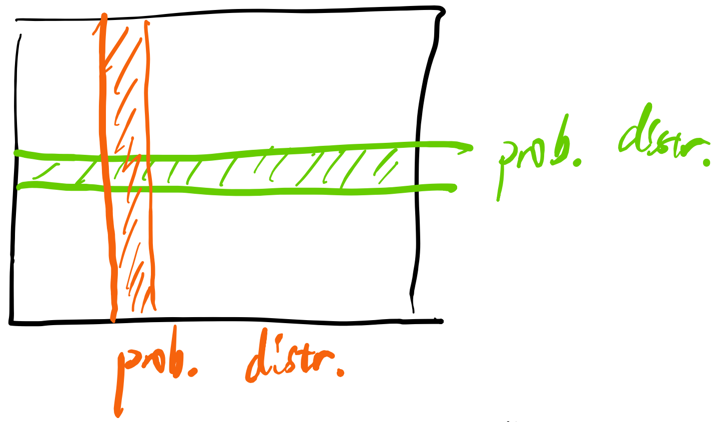
e.g. "Doubly stochastic" matrices.

$P =$ 

$\forall i \in S.$ $\sum_{j\in S} P_{ij} = 1$, & $\sum_{j\in S} P_{ji} = 1$.

Fact. If P is doubly stochastic ($|S| < +\infty$). then uniform distribution is stationary.

(This actually extends to stationary measures when $|S| = +\infty$)

Proof.

$$\frac{1}{|S|} = \pi_i \underset{\checkmark}{\overset{?}{=}} \sum_{j\in S} \pi_j P_{ji} = \sum_{j\in S} \frac{1}{|S|}\cdot P_{ji} = \frac{1}{|S|}$$

eg. SRW in 1-D.

$$\pi(i) = 1 \quad (\forall i \in S) \text{ is a stationary measure.}$$

Def. Reversibility / detailed balance (w.r.t. π)

$$\pi_i P_{ij} = \pi_j P_{ji} \quad (\forall i,j \in S).$$

Fact. If P is reversible w.r.t. π

then π is stationary for P.

(for distributions / measures).

Proof.

$$\pi_j \overset{?}{=} \sum_i \pi_i P_{ij} = \sum_i \pi_j P_{ji}$$

eg frog walk, 1-D SRW.

eg. Biased RW. $X_{n+1} = X_n + \varepsilon_{n+1}$, $\varepsilon_n \overset{iid}{\sim} \begin{cases} 1 & \text{w.p. } p \\ -1 & \text{w.p. } (1-p). \end{cases}$

$$\pi(i) = \left(\frac{p}{1-p}\right)^i.$$

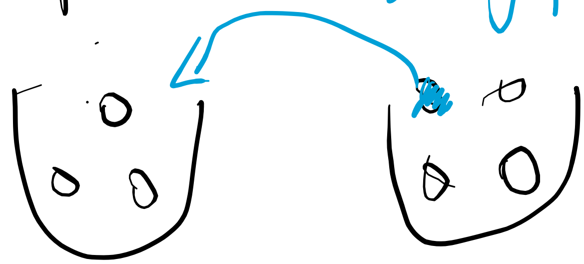
$$\pi_i P_{i(i+1)} = \left(\frac{p}{1-p}\right)^i \cdot p = \frac{p^{i+1}}{(1-p)^i}$$

$$\parallel$$

$$\pi_{i+1} P_{(i+1)i} = \left(\frac{p}{1-p}\right)^{i+1} \cdot (1-p) = \frac{p^{i+1}}{(1-p)^{i}}$$

eg. Ehrenfest Urn.

randomly picked (unif over all d balls)



Box 1 Box 2

X_n = # balls in Box 1 in n-th round.

Guess: $\pi(i) = \binom{d}{i} \cdot 2^{-d}$ for $i = 0, 1, \dots, d$

Intuition: each ball is thrown into a box.

$$\pi_i P_{i(i+1)} = \binom{d}{i} \cdot 2^{-d} \cdot \frac{d-i}{d}$$

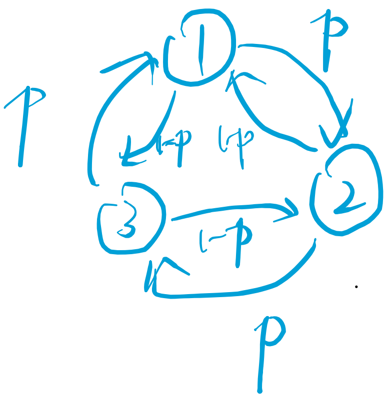
$$= \frac{d!}{i!(d-i)!} 2^{-d} \cdot \frac{d-i}{d} = \frac{(d-1)!\, 2^{-d}}{i!(d-1-i)!}.$$

$$\|$$

$$\pi_{i+1} P_{(i+1)i} = \frac{d!}{(i+1)!(d-1-i)!} \cdot 2^{-d} \cdot \frac{i+1}{d}$$

$$= \frac{(d-1)!\, 2^{-d}}{i!(d-1-i)!}$$

e.g. MC P w/ stationary distribution π but detailed balance is false.



$\pi_i = \frac{1}{3}$ ($\forall i \in S$).

But detailed balance is false when $p \neq \frac{1}{2}$.

Existence & uniqueness of stationary distributions.

Fact. If $|S| < +\infty$, then $\exists$ a stationary distribution.

(Brower fixed-pt thm).

(may not be unique, e.g. reducible MC).

Thm ("vanishing probabilities").

If $\forall i,j \in S$, $\lim_{n \to +\infty} p_{ij}^{(n)} = 0$ then $\nexists$ stationary distribution.

(c.f. transience $\Longleftrightarrow \sum_{n \geq 1}^{+\infty} p_{ij}^{(n)} < +\infty$

So stationary distr. requires recurrence for irreducible MCs).

Proof. Suppose π is a stationary distribution.

$$\pi = \pi P = \pi P^2 = \cdots = \pi P^n = \cdots$$

$$\pi_j = \sum_{i \in S} \pi_i P_{ij}^{(n)} . \xrightarrow{n \to +\infty} \sum_{i \in S} \pi_i \cdot \lim_{n \to +\infty} P_{ij}^{(n)}$$

$$\shortparallel$$

$$0.$$

Contradiction.

Need to justify

$$\sum \lim = \lim \sum .$$

"M-test": For any infinite-dim matrix $\{x_{n,k}\}_{n,k \in \mathbb{N}_+}$

Suppose $\forall k$, $\lim\limits_{n \to +\infty} x_{n,k}$ exists

and $\sum\limits_{k=1}^{+\infty} \sup\limits_{n \geq 1} |x_{n,k}| < +\infty$ Key cond.

then $\lim\limits_{n \to +\infty} \sum\limits_{k=1}^{+\infty} x_{n,k} = \sum\limits_{k=1}^{+\infty} \lim\limits_{n \to +\infty} x_{n,k}$.

(See Rosenthal Appendix for proof. essentially DCT).

Verifying the condition:

$$\sum_{i \in S} \sup_{n \geq 1} \left| \pi_i P_{ij}^{(n)} \right| \leq \sum_{i \in S} \pi_i = 1.$$

Rmk: Justification by M-test is necessary

for SRW, LHS $= 1$, RHS $= 0$. ($\pi(i) = 1$ $\forall i \in S$ stationary measure).

- For finite $|S|$, $\sum_{j \in S} p_{ij}^{(n)} = 1$

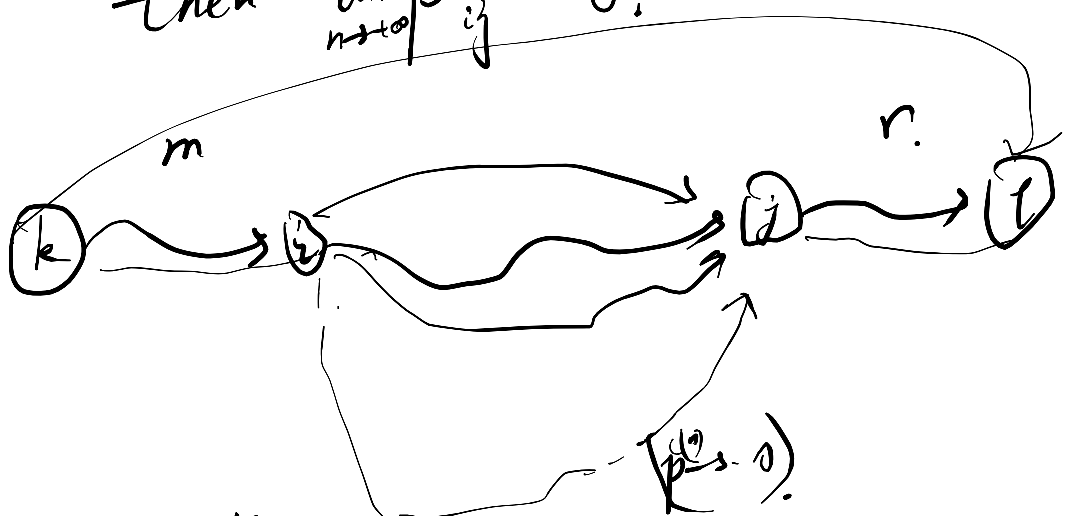
  so $p_{ij}^{(n)}$ cannot all go to 0.

Relaxing "$\forall i, j$" (as in recurrence equiv. thm.)

"Vanishing lemma":

If $k \to i$, $j \to l$, $\lim_{n \to +\infty} p_{kl}^{(n)} = 0$

then $\lim_{n \to +\infty} p_{ij}^{(n)} = 0$.



Proof. $n \ge m + r$

$$\underbrace{p_{kl}^{(n)}}_{\to 0} \ge \underbrace{p_{ki}^{(m)}}_{\text{fixed}, > 0} \cdot p_{ij}^{(n-m-r)} \cdot \underbrace{p_{jl}^{(r)}}_{\text{fixed}, > 0}$$

Fix m, r, $n \to +\infty$

Cor. For irreducible MC, if $\exists k, l \in S$.

$$\lim_{n \to +\infty} p_{kl}^{(n)} = 0 \quad \text{then } \nexists \text{ stationary distr.}$$

Cor ("vanishing together").

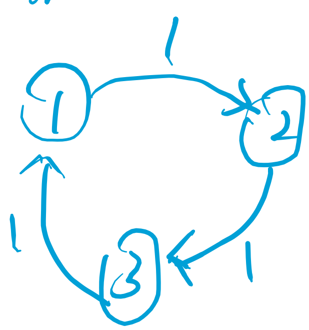
For irreducible MC, one of the following happens.

(i) $\forall i, j \in S$, $\quad \lim_{n \to +\infty} p_{ij}^{(n)} = 0$

(ii) $\forall i, j \in S$ $\quad \lim_{n \to +\infty} p_{ij}^{(n)} \neq 0$.

Note: in the latter case, the limit may not exist.

eg.



$$p_{11}^{(n)} = \begin{cases} 0 & \text{when } \frac{n}{3} \notin \mathbb{Z} \\ 1 & \text{when } \frac{n}{3} \in \mathbb{Z}. \end{cases}$$

But the time average converges

$$\frac{1}{n} \sum_{t=1}^{n} p_{11}^{(t)} \longrightarrow \frac{1}{3}.$$

To avoid pathology above, we define

Def. Period of a state $i \in S$ is greatest common divisor (gcd) of

$$\{ n \geq 1 : p_{ii}^{(n)} > 0 \}.$$

e.g. for the cyclic chain above, period of any state $= 3$.

e.g. If $P_{ii} > 0$, then period of i is 1.

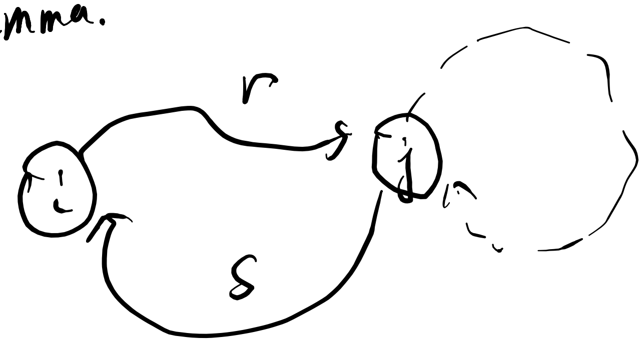
Def. "Aperiodic" $\iff$ period $= 1$.

Lemma: (Equal period).

If $i \leftrightarrow j$ then i, j has the same period.

(Cor. irreducible MC, all states have the same period)

Proof of Lemma.



$P_{jj}^{(n)} > 0$.

$P_{ij}^{(r)} > 0$, $P_{ji}^{(s)} > 0$.

$$P_{ii}^{(r+n+s)} \geq P_{ij}^{(r)} P_{jj}^{(n)} P_{ji}^{(s)} > 0.$$

$$P_{ii}^{(r+s)} \geq P_{ij}^{(r)} \cdot P_{ji}^{(s)} > 0.$$

Let t_i, t_j be periods of i, j, respectively.

$$\frac{r+n+s}{t_i} \in \mathbb{Z}, \qquad \frac{r+s}{t_i} \in \mathbb{Z}$$

So $\frac{n}{t_i} \in \mathbb{Z}$ $\forall n$ s.t. $P_{jj}^{(n)} > 0$.

t_j is gcd of

So $t_i \mid t_j$. By symmetry $t_j \mid t_i$.

So $t_i = t_j$.